A Suite of Computer Programs to Simulate Elementary Aspects of Constructing Tonal Music

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Part 1: Introduction and musical essentials

1 About this article

This article gives an account of my personal exploration of algorithms, as encoded in computer programs, to create and shape simple pieces of music in the western tonal tradition, as exemplified by hymn tunes and short 'classical' pieces written between about 1700 and 1880. I have tried to formulate and apply the 'rules' of melodic and harmonic progression as set out in the many textbooks on musical theory. Rather than a single master program, I have built a suite of a dozen or so complementary programs, each dealing with a specific aspect of tonal music. Together they could form a toolbox for the user to apply in the correct order while using their judgement to improve on the program's output, which inevitably will be crude. Each program simply uses random numbers to select options from amongst the musical possibilities allowed by the rules, though in most cases the user can overrule this and make their own choice. I emphasise that I have been concerned only with the basic carpentry of music, not the achievement of worthwhile composition.

Many people more competent than me have explored the interface between musical composition, computers and artificial intelligence. There is the Society for Mathematics and Computation in Music which holds international conferences and publishes a regular learned Journal of Mathematics and Music. The internet has accounts of computers being used to analyse harmonies and melodic lines in musical common practice, particularly Bach's chorales. Computer invention of music has been demonstrated by, for example, David Cope (*artsites.ucsc.edu/faculty/cope*), professor of music theory at the University of California. He has produced several advanced programs which generate quite convincing music in the style of Bach's time, some of which has been published on a commercial CD. On 21st March 2019, the attributed anniversary of J. S. Bach's birth, Google's web page opened with a banner in which the user was invited to enter a few notes and have them harmonised in the style of Bach, using artificial intelligence in real time. My efforts are far less sophisticated and the results far more modest. The text books from which I gleaned the 'rules' include 'Tonal Harmony in Concept and Practice' by Allen Forte, one of the best, and also 'Tonal Harmony' by Kostka and Payne. Also useful and well written is Thomas Benjamin's 'Counterpoint in the Style of J S Bach'. Where textbooks and literature searches could not provide quantitative data, I carried out my own limited analysis of musical samples – for example, to determine the probability of any chord X leading to chord Y.

I wrote an embryonic suite of computer programs in the early 1980s as an aid to learning the theory of harmony. At that time the memory of home computers was very small – only 32 kb – so programs had to be short and compact. I wrote them in BBC BASIC to run on the Acorn BBC micro-computer. Since then an excellent simulation of BBC BASIC has been developed for Microsoft Windows by Richard Russell (*bbcbasic.co.uk*), and I have continued to use this language even though powerful languages suited to artificial intelligence are now available. I am therefore revisiting old algorithms which are essentially rule based. I will not publish the programs themselves until I have improved the coding and user interface, both of which are currently crude and make-shift. So this article presents only outline descriptions of the algorithms and examples of their outputs.

I have numbered the programs in the order they were created, which in general is the order of increasing complexity and musical detail. As stated above, they form a toolbox by which the user may generate musical material under his or her limited control. This material will then have to be organised by the user since the programs do not attempt to create complete musical works. There is no orchestration, so the user must have instrumentation in mind when setting the option flags in each of the programs.

There are 12 main programs in the suite and they naturally divide into three groups:

1. Group 1: programs dealing with harmonic structure, which produce a 'hymn tune'. This is a musical skeleton in the form of a progression of chords with correct voice leading from chord to chord, and with assignment of cadences and note durations within a given time signature. Programs 1, 6, 4, 5, 7 and 8 are in this group and would be used in that order to create a hymn-tune like piece in 3 or 4 time. They are described in §3 to 10.
2. Group 2: programs which add surface decoration to or derive variations from such a hymn tune, so embellishing melodic lines or creating a tune and/or texture. Programs 9, to 12 are in this group. These are described in §11 to 16.
3. Miscellaneous. Three programs stand alongside the above Groups 1 and 2. Given a note or notes, Program 0 will identify those chords in C major or c minor to which they could belong, and a variant of Program 0 will identify the key of a given piece. Program 2 will create a single, simple, purely vocal line, a *cantus firmus*, a feature of music from an earlier age of church singing. Program 3, described in Appendix 1, is a set of file conversions which convert the output from either of these groups into Lilypond files for notation as sheet music and as a MIDI file for immediate audio playback. Programs such as Sibelius or Steinberg's Dorico could be used to elaborate and orchestrate the piece.

Below is an overview of topics in each section of this article.

- §2 gives an overview of the essential ideas behind tonal music. I set these out because they have been my guide in developing the algorithms.
- §3 opens Part 2 of this article on the Group 1 programs by presenting a considerable body of data on the frequencies of occurrence of various chords and chord sequences. This is the foundation of the 'hymn tunes' upon which all programs are built.
- §4 describes how I have used this data to build Program 1, which generates a string of triads in either the major or minor mode, labelled only by Roman numerals, in an order which makes

musical sense. A sub-section discusses modulation to the dominant and relative minor keys. Using Program 3 these chord names can be translated into block chords and notated and played as MIDI files in Lilypond.

- §5 is about voice-leading within a single voice/melodic line. It describes Program 2 to generate a *canto fermo* melodic line in notes of equal duration. Subsection §5.3 discusses options for taking forwards the music-generating programs. Mention is made of Program 3 which converts the output of the note-generating programs into a Lilypond file (*www.lilypond.org*) from which it is engraved as a musical score and converted to sound as a MIDI file. Program 4 is a development of Program 2 in which the shape of the melodic line is controlled by a user-defined curve. I call these voice-line guide curves.

- §6 describes stages in the development of Program 5 which makes a first attempt at 4-part SATB harmony. The approach is to lay four voice-line guide curves across the sequence of triads generated by Program 1. Once preliminary measures to prevent parallel unisons, octaves and fifths have been put in place, the results are fair but not perfect. A subsequent program, Program 7 of §9, corrects most of the errors.

- §7 is a diversion from computer codes. It discusses several aspects of rhythm and the development of melody with rhythmic interest. Subsections examine agogic accent in polyphonic music, cadences and some aspects of sequences, This section effectively defines the challenges and sets out the agenda for developing further algorithms within the Group 1 programs.

- §8 outlines Program 6 with examples. This is a first step towards imposing form on the sequences of triads, following the discussion in §7.2. On the assumption that a piece has a given time signature and is constructed from a number of 4-bar blocks, Program 6 searches for chord pairs which could become cadences at suitable positions through the sequence, allocates a number of beats according to a user-selected time signature, and then allocates the other triads to beats between the cadences. The output is in an internal code, but can be realised as a piece of music using Program 3.

- §9 returns to the challenge of producing 'correct' 4- and 3-part harmonisation of a sequence of triads. Program 7 attempts automatically to correct the faults in this first version by completing all triads with root, 3rd and 5th, and to correct any remaining parallel unisons, fifths and octaves. It builds upon Programs 1, 6 and 5, firstly applied in that order, to generate a crude first harmonisation. The corrections of Program 7 are done using specific methods and not by a tree search for improvement. At its final stage Program 7 scans the whole piece to identify remaining weaknesses in voice progression. Rather than attempt to improve all of these automatically, it lists the faults on screen and invites the user to make a few by-hand adjustments, either using the simple note editor which is part of the program or with the Frescobaldi editor for Lilypond.

- Program 8, described in §10, inserts a linear (scale-like) sequences between suitable chords in the 'hymn tune'. These so-called 'linear intervallic patterns' (Allen Forte's label) extend the piece without disturbing its fundamental harmonic flow. They thereby break up the block-like musical structure by creating sections of unequal length and so add interest and tension, since they delay our arrival at the anticipated cadences. They are a very common feature of music from 1700 and later. That concludes description of the Group 1 programs.

- §11 opens description of the Group 2 programs which deal with melodic surface detail. Whereas §7 deals with the skeletal and muscular structure of music, §11 deals with the surface detail – musical motifs, compound melody and sequences, and the textures of 18th and 19th century keyboard and piano writing. It also briefly considers the key question "What makes a good tune?". My original investigation to answer this has been published in a separate article on *mathstudio.co.uk* under that title.

- §12 is a discussion of motifs, compound melody, ornamentation and other stylistic features of 18th and 19th century western music.
- The specific topic of inserting passing notes is accomplished with Program 9, described in §13.
- §14 describes Program 10 which decorates a single voice line by allocating rhythmic and pitch patterns in auxiliary notes to each bar. An auxiliary note is one for which the pitch varies by only ± 1 diatonic step from the given note, since steps of a 3rd or larger would imply a triad. This section contains the first example of a short complete piece of music, and a recording played on a church organ of this is on the author's web sit at *www.mathstudio.co.uk*.
- §15 describes Program 11 which creates single musical lines made of broken chords, derived from a given bass or treble voice line plus the sequence of triads to which that line fits. The results for bass may sound like the left hand accompaniments of piano pieces. Those in the treble can sound like Lisztian decoration or perhaps like a lively dance tune. Many variations are readily generated, so the user must make a selection according to the type of piece and instrument in tended.
- Program 12, §16, creates melody by splicing two or more given vocal lines into a single line. This section quotes musical examples of arpeggiation, octave jumps within a voice, pedal notes and compound melody.
- §17 takes stock of where we have got to. It presents some examples of short pieces of music 'composed' using programs in the suite.
- Appendix 1 outlines conversions of note nomenclatures with Program 3. Appendix 2 discusses the way we determine the key and time signatures of a given vocal line and outlines an ancillary program, Program 0, which attempts to do this.

I come back to the point that these programs deal only in the elementary procedures in the carpentry of musical composition. What is lacking is that overall intelligent, tasteful sense of style and integrity which gives a piece a definite character. While we cannot see inside the mind of a fine composer, we may suppose that they start with some purpose for their new piece, and have in mind its character, what instruments it is for, its length, even its key. There is none of this guiding the algorithms in my suite of programs. That is why they can be no more than a toolbox, generating output which may prompt a fertile musician who might otherwise have composer's block – a spur to innovation, but certainly not an end in itself. That said, there has been much inferior music produced over the centuries by jobbing musicians – music with no definite character, immemorial tunes, pedestrian harmony, no rhythmic interest, which go on too long. Indeed, classical recording companies, in their search for something new to record, have dredged up many hours of third rate stuff which you don't want to hear again. I say this not to disparage musicians of limited talent who somehow had to make a living, but to remind us just how rare and precious great, enduring music actually is.

2 Concepts and principles of tonal music

This section is an overview of the basis of western classical music. It sets out the material and themes which the computer programs deal with.

Western tonal music is based upon the following inter-related concepts:

1. Fundamentally music is rooted in our common human nature. Its essence is in i) the rhythm of breathing, heart beat, in singing, walking, dancing, and ii) in our sense of place – here, there and return: something familiar, something strange.

2. Classical music has the feeling of motion in time as the piece is played, from an initial state from which it departs, through some intermediate states, until it returns to final rest close to where it started. The sense of moving from a stable, familiar aural state to an unstable, different one and then to another stable state drives tonal music over short, intermediate and long time spans.

3. Stability depends essentially on familiarity. Repetition is key to this. Repetition of a tune, even a short motif, a rhythm or a harmonic chord creates familiarity and hence some degree of understanding of where we are. In contrast, ever changing melody, rhythm or harmony creates unfamiliarity, the sense of a journey, of something unfolding, or perhaps, if so very different, of incomprehension and alienation. Too much repetition is boring, tedious, even irritating. Too little leaves the listener bewildered. The control of repetition, on both short and long time scales, is a crucial aspect of musical construction. The long term organisation of repetition combined with new material is the essence of musical form.

4. From the harmonic point of view, musical excursion is measured in terms of the musical distance from a tonal centre called the tonic. Simple pieces always end at the tonic, with the tonic note as the last note in the bass, and almost all also start at the tonic, or quickly arrive at it after a short preamble. This contrasts with modern so-called minimalist music where there is cyclical repetitions of short musical ideas with little feeling of departure and return. Background music to some films, TV dramas and documentaries can have a similar sense of existing only in the moment, deliberately devoid of either beginning or ending.

5. Musical sounds are constructed from notes, each of which has a pitch and a duration. Pitch is our human perception of acoustic frequency. We perceive notes whose frequencies are integer multiples of each other as being in a musical sense essentially the same; such notes are one or more octaves apart and so we give them the same reference letter as name. Thus 'C' is the family which includes middle C, denoted C4 with frequency $261 \cdot 6$ Hz, and C5 an octave above with frequency $523 \cdot 2$ Hz. It also includes the lowest note on a double bass, a 'cello and a viola. The frequency ratio of $3/2$ defines the musical interval of a perfect fifth and $5/4$ the major third, ratios first investigated by Pythagoras. When a string is plucked or a rectangular elastic plate struck, the string or plate will vibrate in a pattern of natural modes at frequencies in these simple ratios. The three musical intervals of octave, perfect fifth and major third which constitute the major consonant triad fundamentally arise from our perception of the sound radiated from simply shaped vibrating objects.

6. Historically western music developed largely from courtly and folk dances, and from unaccompanied church and folk singing. It was held in aural memory and passed between generations before ways of notating it were devised. In the music with which I have been concerned, the vocal nature remains even when written for instruments which can play unsingable pitches and

unsingable intervals. The smallest pitch difference which is notated is the semitone, which is about one twelfth of an octave. Indeed, in the equal temperament system of tuning, widely adopted since Bach's time, a semitone is defined as exactly 1/12 of an octave, corresponding to a frequency ratio of $2^{1/12} = 1 \cdot 0595$, that is, about a 6% increase or decrease in frequency. The interval of a fifth is 7 semitones, which on the equal temperament scale is a frequency ratio of $2^{7/12} = 1 \cdot 493$, not the exact $1 \cdot 50$ of a true perfect fifth. The minor third is 3 semitones and the major third 4 semitones.

7. If all the gaps between fundamental, third, fifth and octave are filled in with semitones, we obtain the chromatic scale. Chromatic scales, because of their equal spacing, have no singular or prominent pitches, so cannot define a tonal centre. Omitting selected chromatic notes gives weight to the others and removes intervals which are difficult to sing, such as 6 semitones (C to F♯). Several centuries ago the singable pitch sets formed the six ecclesiastic modes, but since the 17th century all but two have fallen into disuse, leaving the diatonic major scale (the Ionian mode) and minor scale (from the Aeolian mode). In the minor scale the 6th and 7th degrees can be raised a semitone to form the familiar ascending and harmonic minor varieties. The basis of tonal music is the set of triads shown in Figure 1, whose roots (lowest notes) are each of the degrees of the major and minor scales. These are the entities I have been working with

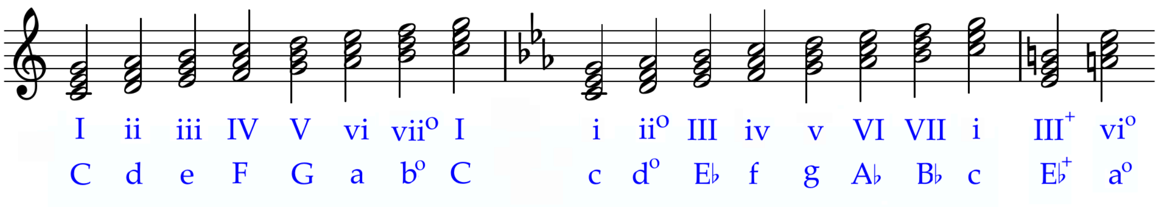

Figure 1: Major and minor triad scales based on C as tonic. Capital letters denote major triads, lower case ones minor triads, ° denotes a diminished triad, and + an augmented one.

in the computer algorithms. Clearly, any of the 12 chromatic pitches can be used as the tonal centre, with the triads in Figure 1 merely transposed as a set through the requisite number of semitones from C. Thus we refer to a piece being in D or A♭major, or in d or f minor. Nevertheless, in pieces nominally in the major there is much borrowing from the minor mode and *vice versa*.

8. The movement in time of much tonal music is against a generally steady succession of beats arranged aurally into bars of usually 2, 3, 4, 6 or 8 beats. We can think of the clock-like tick of beats and bars forming a framework of narrower and wider periodic marks in time on top of which the melody, harmony and rhythmical embellishments are placed. This sense of beat seems fundamental to us because of our heart beats, regular breathing, and the repetitive left-right left-right while walking and running, and the down-up-up down-up-up of dancing. Beats that fall at the start of a bar are felt to have more stress or accent, so we perceive strong and weak beats, as in poetry. The harmonic rhythm of a piece refers to how the harmony changes in relation to the beat, and correspondingly the melodic rhythm. If both melody and harmony change with each beat, the piece can sound like a hymn tune. Often, however, the melody changes more frequency than the harmony; perhaps the simplest example of this is an arpeggiated chord as shown in Figure 2.

9. Much of the forward movement of tonal music is created by melodic progression from one pitch to its nearest neighbour in the same scale. Think of one person singing on only notes from a major or minor scale. The smoothest line, and probably the easiest to sing, has stepwise

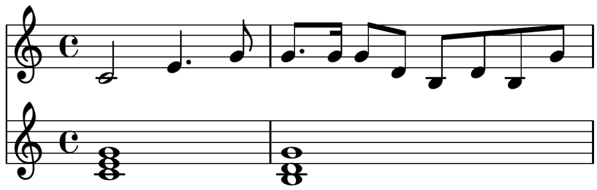

Figure 2: Melodic rhythm (top line) can be more rapid and varied compared with the harmonic rhythm (bottom) which it elaborates.

movement in which successive notes are only one or two semitones apart. If any particular note has as its neighbouring notes one which is one semitone distant, say above, and the other neighbouring note two semitones lower, aurally we feel drawn next to choose to sing the note closer in pitch rather than the one two semitones away. For instance, in the C major scale, singing B will usually make us want to sing the C above next rather than the A below. Similarly an F will tend towards the E below rather than the G above. Forte calls this the 'law of the half-step', and we might think of it as an aural 'magnetic attraction'. You need will power to sing an ascending major scale and stop at degree 7; you feel compelled to go on to complete the scale by singing the tonic octave. For this reason, degree 7 is called the 'leading note'. (This is the sharpened degree 7 in the minor scale.) For the same reason the A♭ in the c minor descending scale falls naturally onto the G below. This voice leading by steps of 1 or 2 semitones is the primary means by which one triad is linked harmonically to the next. The melodic progression of voices and the harmonic progression of chords are intimately tied.

10. Much of tonal music developed from singing of three- or four-part choirs, with boy's or women's voices in soprano and alto and men's in tenor and bass – the SAB and SATB choirs. This transfers into chamber music as, for instance, the string trio of violin, viola, cello and the string quartet. Each of these 'voices', whether human or instrumental, follows its own melodic line and together their simultaneous sounds create the harmony. After the soprano line, the bass is the next most important. It can move stepwise, but also is powerful when it moves by jumps of a perfect 5th, or its inversion, the perfect 4th. Aurally this probably is because in each note we perceive not only the fundamental pitch, C4 say, but also the first and second harmonics which are an octave above (C5), and an octave plus a fifth (G5), respectively. The 4th and 5th are singable intervals. When soprano and bass sound together, moving on each metrical beat by steps and/or by 5th in the bass, there is the strongest sense of forwards musical momentum. Figure 3 gives some examples.

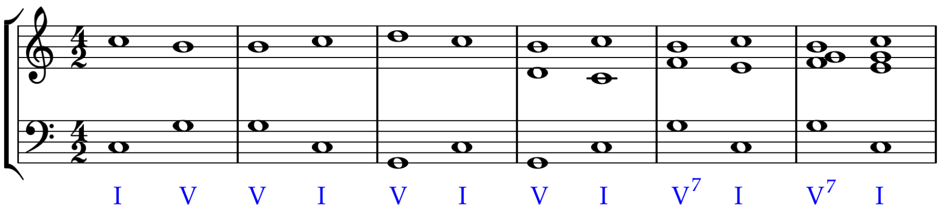

Figure 3: Chords formed by stepwise voice leading and movement of the bass in 5ths.

In this figure the first bar is an opening progression from the tonic (I) to dominant (V); we hear this even though the triads are not complete. Bars 2 and 3 are the reverse, a V-I progression familiar as a perfect cadence. At bar 4 both triads are complete and in the final two bars the 7th is added to V; this resolves by falling through a semitone to the third of the tonic chord.

The combined progression I - V - I forms the musical axis of almost all tonal compositions. As explained in §7, in practice it is elaborated on the small scale by passing notes, auxiliary notes, suspensions and rhythmic devices, drawn out on a larger scale by inserting chords which lead away from I and prepare the ear for V or I, and on the largest scale by modulation to related keys, particularly the dominant. Nevertheless, I - V - I remains the underlying logic and forms the overarching aural experience; we depart from I, visit V and return to I.

11. Much music, like speech, proceeds in phrases, sentences and paragraphs, punctuated by pauses of different lengths. If we think of a person reading aloud or speaking in public, the pauses are brief between phrases and longer between sentences, so that the listener has time to take in the points just been made. A change of paragraph means a change of topic or focus, and is signalled by a longer gap and often a few words of introduction. Composers of music must group and order their thoughts and punctuate their delivery if their message is to be intelligible. Similarly performers must make the phrase structure clear to the audience. In music the places of pause are the cadences, a word derived from the Latin word meaning 'to fall'. It aptly describes both the typical fall in pitch and the halting in pace of the music. The opening I - V and closing V - I triad pairs form the most effective and common cadences. In each the melody will usually pause on the second of the pair.

12. Within the scale of C the continued rise of a 5th, or fall of 4th, from tonic C takes us in turn to the notes G, D, A, E, B, F and back to C. If triads are erected on these roots, there is a pull from the G to move back to C (V to I) and similar but weaker pulls from d (d minor) to G (ii to V), a to d (vi to ii), e to a (iii to vi), b^{o} (diminished) to e (vii° to iii) and C to F (I to IV). These are often arranged into a chain of triad-based chords such as I vi ii V I, which is very common. The minor triad vi shares the two pitches C and E with triad I, so the chord progression I to vi is a small change. vi then prepares the ear for ii, ii for V, and V drives to I. Since IV shares two pitches with ii, it often substitutes for ii as a so-called 'dominant preparation' triad, so I IV V I is another common sequence. The anomaly is iii for though iii also shares two pitches with the tonic (E and G) it sounds quite remote from I. Perhaps this is because it contains the leading note, which is remote from the I triad. In practice the progressions I - iii and iii - I are rare.

13. Old folk songs and church chants were sung unaccompanied and were usually based on the ecclesiastical modes, especially the Dorian (notes D E F G A B C D) and Phrygian (E F G A B C D E). Since about 1700 melodies have had a tonal basis whereby the melody implies a harmonic background and develops within the logic of triad progressions such as I vi ii V I. In this way harmonic progression, itself largely driven by voice-leading from chord to chord, reacts back on melody to guide the overall melodic progression over several bars. The intervals of 3rd between root and 3rd, and 3rd and 5th of the triad are often filled with sub-metrical passing notes to give smooth movement in 1 or 2 semitone steps. The listener expects the salient notes of the harmony to fall on the stressed beats of the bar. The harmony notes may be decorated with upper or lower auxiliary notes – notes a musical 2nd above or below. The better melodies have some overall shape. An arch is popular, rising to a single highest note, often on the chord V, then falling to the closing V - I cadence. Any shape which is over repetitive or just meanders around the same note usually sound poorer in interest and less memorable. Another important melodic construct is the sequence in which a recognisable rhythmical and melodic phrase is repeated two or more times at different pitches, usually one scale note lower or higher than the previous. The sequence as a whole will have a harmonic goal, but while a long sequence is being played the normal 'rules' of harmonic progression are mostly suspended because the music is driven by the powerful scale-like melodic line.

14. When four voices are in play, two of them must share the same note of the current triad. There are rules, based on practical experience, about which note should not be doubled, largely brought about by the desire to avoid parallel octaves or perfect fifths between any two voices. Such parallels stand out to the listener as moments when two voices seems to reduce to only one, giving the harmony a sudden emptiness – a hole in it. The general rules are that the leading note 7 of the scale is hardly ever doubled since both 7s would tend strongly to rise by a semitone to the tonic. Similarly any dissonant note (one that does not belong in the prevailing triad) is not doubled because both would seek to resolve in the same way. One example would be the 7 in V^7; another is any suspended melody note. Dissonant notes and chords are not necessarily unpleasant to the ear, but rather that the sound feels strained and unstable, with a need to topple forwards to some resolution of its internal tension. The accumulation and subsequent release of aural tension is the essence of movement in tonal music.

15. Genuine music, unlike the elementary building blocks discussed in this article, has an expressive purpose framed within an integrity of style and content, the creation of a trained and talented person. Except for convenience in discussion, melody, rhythm, harmony, counterpoint and form are inseparable, and together drive the music forwards on a short timescale. On a larger scale the forwards motion in many pieces is produced by a contrast of themes and by movement from one key centre to another through the processes of modulation. Often a new key is emphasised by a new theme or variation of a theme, or a change in dynamics. The so-called 'development' or elaboration of ideas is akin to the developing complexities in a play or novel as the characters interact with one another and surprising events occur. Drama, emotion and wit can be portrayed at the highest level. Producing a convincing piece requires the highest ability, and is certainly beyond the scope of any computer program which I could write.

Part 2: Programs in Group 1: Harmonic structure

3 Musical sequences of triads

The programs in Group 1 deal with harmonic structure, for which the sequential movement from one chord to the next is fundamental. A starting point, therefore, has been to document evidence on the chords progressions used over the period 1700 to about 1850, but not including the chromatic elaborations of later composers. We will be concerned both with the frequency of occurrence of individual chords of the diatonic major and minor modes, and with the ordered pairing of chords. I have not sought sequences of three chords *per se*.

When I started work on these programs some 40 years ago, I made judgements about likely chord progressions using the instruction in harmony textbooks. For instance, V often follows I so the probability of I - V might be about 50%, while V hardly ever follows iii so the probability V - iii might be judged to be less than 5%. Since then some quantitative statistics of the chord sequences have been published and I have made my own limited survey. This section gives a brief account of these enquiries.

In the fairly recent literature is the thesis on modelling Bach chorales with Markov chains by Michaela Tracy, Harvard University, 2013. Her work seems fairly close to mine but more sophisticated and effective. She reports that her computer model produced chorales which musical experts could not readily distinguish from Bach's own.

In reading sheet music, even when 7th and other non-triad notes are absent, assigning a Roman numeral to a chord is not straightforward where modulation of key may be occurring; there can be ambiguity in identifying a chord's function whenever music modulates. For example, consider a sonata form first movement which starts in C major but by the end of the first section has clearly modulated to the dominant key, G major. Do we label an *a* minor triad as vi in C or as ii in G? Some authors deal with this by giving the notation in both keys in parallel. What a listener will consider to be the prevailing tonic will depend on how much time has passed in the new key for it to sound familiar and hence stable. Forte points out that a listener becomes aware that a piece has modulated to another key only after that key has become stabilised over several bars. Short-lived colouration by, say, sharpening the 3rd of a minor triad to form its major may have been called 'modulation' in older textbooks, but would not now be regarded as such. The issue looms large in minor key pieces since they often veer to the relative major after only a few bars. In the literature the various researchers seem to take different stances on what is the locally prevailing key, and this brings some uncertainty when comparing their numerical values of chord frequency.

My own view is that in chorales and other short pieces, particularly of the baroque era, there is not time for a new key to be established and I would therefore refer all triads to the tonic of the piece as given by the last note in the bass. Indeed, I consider Bach's two-part inventions, movements of the keyboard and unaccompanied cello suites, and the preludes in 'The 48' each to be in a single key. Modulation to stabilise another key was not a strong part of baroque styles even though in allemandes, gigues, courantes, etc. with an AB binary structure, the A section ends with a cadence onto the dominant triad preceded by II with a sharpened 3rd. In the classical era, however, modulation to the dominant or relative minor keys, emphasised by introduction of new themes, was a distinctive part of the newly developed extended musical forms, particularly sonata form. Therefore with these works chords would best be referred to the deliberately established current tonic. When chord pairs are being analysed, the matter is somewhat easier in that there is less ambiguity in determining the interval between the roots of the two triads.

3.1 The occurrence of individual triads

'Frequency' here means frequency of occurrence, not musical pitch. In §4.1 we examine evidence for the relative occurrence of individual triads, then in §3.2 address two triads in succession. I have considered four data sets found in the internet literature and also carried out my own data collection on short pieces in major and minor keys.

David Temperley, a professor at Eastman School of Music, has carried out a statistical analysis of chords and harmonic progressions, (*www.davidtemperley.com*, 2009). The data set was 48 short musical excerpts in major and minor keys quoted in the textbook by Kostka and Payne, containing 919 chords and lasting 1354 seconds. The excerpts included composers from Bach to Tchaikovsky. Table 1 gives the relative occurrence of each chord type in descending order as determined by its root relative to the tonic. 35% of chords are tonic, I, and 23% dominant, V. If the diminished vii° is counted as a quasi-dominant, this goes up to 27%. Some of the tonics would be Ic which is really a type of V. The so-called dominant preparation chords, ii, IV and vi, occur in that order of frequency and together represent 24% of all chords. iii and other triads are rare in major. (Lower case means that the triad is minor.) Comparing the durations with the number of chords, we see that music dwells on the tonic, while ii is passed over rather quickly. This all accords with traditional teaching. Perhaps the only surprise is the short time on E♭, the relative major of c minor. Table 1 is consistent with the findings of Rupert Deese (*rupertdeese.com/resources/ai_final_report.pdf*) who analysed all 371 Bach chorales and found that the most common chords are I, V, V^7, IV, vi, ii^7 and ii, though he does not give relative frequencies.

A second study is by Martin Rohrmeier in his M. Phil dissertation, 2005, published by Darwin College, Cambridge, UK. He analysed the Bach chorales and found the relative frequencies of the

semitone distance	Roman root	root	% by number	% by duration
0	I	C	35	41
7	V	G	23	22
2	ii	D	11	9
5	IV	F	8	7
9	vi	A	5	6
8	♭VI	A♭	4	3
11	vii°	B	4	3
1	♭II	D♭	2	2
4	iii	E	2	2
6	♯IV	F♯	2	1
3	♭III	E♭	1	1
10	♭VII	B♭	1	1
	others		3	2

Table 1: Relative occurrence of triads in 18th and 19th century music according to Temperley. C is taken as the reference tonic. Column 1 lists the distance of the triad root from C in semitones, column 2 gives the triad in Roman notation and column 3 by pitch letter. Column 4 is the percentage number of chords and the last column is the percentage time dwelling on each chord.

most common triads and 7th chords. Table 2 lists the 12 most common in major and minor modes, taken from his table 5.1. The columns for major both sum to only about 60%. Some 7th and 9th chords are listed which sum to about another 10%, but the final 30% are not identified. In the minor column the high occurrence of E♭ and B♭ is striking, but such presence of the relative of c minor is to be expected.

MAJOR Roman	Triad	Frequency %	MINOR Roman	Triad	Frequency %
I	C	18·5	i	c	13·9
V	G	12·1	III	E♭	8·0
IV	F	6·1	V	G	7·2
vi	a	5·5	♭VII	B♭	5·8
V^7	G^7	4·7	iv	f	4·8
ii	d	3·7	V^7	G^7	3·6
iii	e	2·3	♭VI	A♭	3·4
ii^7	d7	2·1	v	g	3·0
vii	b°	2·0	ii°	d°	2·3
vi^7	a^7	1·9	IV	F	2·0
$II\sharp^7$	D^7	1·9	$bVII^7$	$B\flat^7$	2·0
II♯	D	1·9	I	C	2·0
			ii°	d°	1·9

Table 2: Relative frequency of occurrence of the most common chord types in Bach's 371 chorales as determined by Rohrmeier. Lower case means a minor triad and $^\circ$ a diminished one.

Do Rohrmeier's frequencies tally with Temperley's? Direct comparison is not possible because Temperley does not separate major and minor modes, nor does he identify 7th chords separately from the base triad. Also his historic period includes the 19th century whereas Rohrneier is just the Bach chorales. To make some comparison, I have assumed that all chords not in Table 2 have roots in proportion of the chords in Table 2. I have therefore scaled up the frequency values to make the sum of the columns 100%, and also combined 7th chord with their triad bases. The values then transform to those in Table 3, which compare passably well with Temperley's. Broadly, about 30% of chords have I as root. In major key passages about 25% have V as root, about another 25% have IV, ii or vi. The two studies differ most in describing the minor key. Rohrmeier's data is that in the minor III and VII occur in about 13% of roots, with V down to 18%.

MAJOR	%	MINOR	%
I	30	i	27
V	27	♭III	13
IV	10	V	18
vi	12	♭VII	13
ii	15	iv	11
iii	4	♭VI	6
vii°	3	v	5
		ii°	7

Table 3: Data by Rohrmeier in Table 2 scaled and combined to make it more comparable with Temperley's data in Table 1. The frequencies add to 90%.

semitone distance	Roman root	letter root	Mearns Major	White Major	Rohrmeier Major	Mearns Minor	Rohrmeier* Minor	Temperley combined	JMC Major	JMC Minor
0	I/i	C	22	29	30	25	27	35	34	27
7	V /v	G	22	29	27	20	18, 5	23	31	14, 7
2	II/ii	D	16	11	15	14	7	11	11	4
5	IV/iv	F	9	11	10	11	11	8	16	11
9	VI/vi	A	12	8	12	7	11	5	14	
4	III/iii	E	7		4	5		2	7	
11	VII/vii	B	6	4	3	6		4	3	
3	bIII	Eb	1			4	13	1		12
10	bVII	Bb	1			3	13	1	1	14
8	bVI	Ab	2			3	6	4		4
6	#IV	F#	3			3		2		
1	bII	Db	1			1		2		
	other							3		

Table 4: Comparison of relative frequency estimates of triads by root in % according to Mearns, White, Rohrmeier (scaled as Table 3), Temperley and the author (JMC). Column 3 gives the triad root transposed to C as reference tonic. The two values for V in Rohrmeier's and JMC's minor columns refer to the major triad V and the minor one v respectively.

A third set of data is found in table 5.6 of the PhD thesis by Lesley Mearns, University of London, 2013. She analysed the 24 preludes in Bach's Well Tempered Clavier Book 1 and gives in her table 5.4 the root scale degrees of all her hand annotated chords relative to the main key of the prelude measured, as with Temperley, in semitones from the tonic. Yet a fourth set of frequency values is found in table 1 of the PhD thesis by Christopher White, Yale University 2013, at his example 17, p221. White also gives in his table 1, p 244, a comparison by era, in 50 years intervals, of the relative frequency of chords. The dominant 7th supplanted the plain dominant triad after 1800. We do not need to go into this further, however, interesting though it is.

My own data, described in §3.2 below, was collected principally to determine the frequencies of various chord-to-chord transitions, but it also gives as a by-product the relative occurrences of various triads. Table 4 collects all the values from Mearns, White, Rohrmeier (scaled), Temperley and myself (JMC). The triads are arranged roughly in descending order of frequency. The columns for Mearns give the average values she reports over the 24 preludes. Bear in mind that Temperley's are drawn from a much wider time span over which harmonic styles broadened. The much higher frequencies found by Rohrmeier for the E♭, B♭ and A♭ triads may be an example of the problem of identifying Roman numerals when the music modulates to the relative minor. Mearns states that she refers chords to the nominal key of the Bach prelude. Rohrmeier used a 'sliding window' covered several chords and selected the key of the chords local to the current window position. The author's data are on whole pieces nominally in a major or minor key, without any correction; they were short enough that modulation to another key was not really meaningful. My own data agree quite well with the scaled data by Rohrmeier. The largest discrepancy is the frequency of ii or II in the minor between Mearns, Rohrmeier and myself.

Statistics have also been collected on the relative frequency of chords by type – major, minor, diminished, 7th, etc – in some of Bach's works irrespective of which scale degree they fall upon. In Mearns' thesis her table 5.4 gives the frequencies for the major and minor key preludes in WTC I, and in Example 16 of his PhD thesis, Christopher White gives a pie chart aggregating all of Bach's chorales (his example 16, p 219). For the preludes in major keys Mearns gives 38% major triads, 20% minor, 20% dominant 7th, 8% diminished, 6% minor 7th. In minor key preludes her values are 29% major, 28% minor, 18% 7th, 8% diminished, 3% minor 7th. White gives combined values of 42% major, 21% minor, 10% dominant 7th, 6% minor 7th. I consider this quite good agreement.

All the above data can be used to check the plausibility of the output from my own triad-generating program.

3.2 Pairs of chords

We now turn to the probability of one triad leading to the next, which underpins my algorithm. I have used three sets of data:

1. by Temperley, in this internet article,
2. figure 7.1.6 in the book 'A Geometry of Music' by Dmitri Tymoczko, OUP, 2011,
3. my own original data collection on a selection of Bach chorales, English hymn tunes and short pieces.

Tracy gives some tables on individual Bach chorales in her thesis but I have not used them.

Table 5 is taken from Temperley's web page and shows the relative frequency of two-triad progressions in the Kostka and Payne excerpts from the 18th and 19th centuries. They are expressed as percentages of all chords following from the first, arranged by row (first chord) and then column (second). As examples of the meaning of the table, in 38% of occurrences of I (but not Ic) the next chord was V; and V moved to I in 85% of cases. There is some inconsistency in Temperley's table because corresponding rows and columns do not sum to the same values. The cells with white background are probably mostly from passages in a major key, and the blue ones minor. The grey column probably includes diminished 7th and ones with raised 3rd such as V or V (D F♯ A C). The ♯IV row shows strong movement to V. Each row sums to 100%. The bottom row counts the percentage of chords of that type in the sample, which was 755. The reason this differs from the 919 chords used for Figure 1 is that Temperley has identified all the Ic 6 4 chords and counted them as V. The table shows strong evidence for the transitions vi - ii - V - I and IV - V - I in major and B♭ - E♭ - A♭ in the minor/ relative major.

Using Temperley's tables I have applied some judgement to extract the relative probabilities of progressing from one triad to the next in major and minor key passages. The results are given in Table 6. I have combined some columns as indicated because my program treats then as the same. For instance, I combined ♯IV with II, and Ic with V.

The study by Tymoczko seems to have been initiated by him to evaluate a model of harmonic progressions described in chapter 7 of his book, 'A Geometry of Harmony'. He says that one part labelled 3000 progressions in 70 Bach chorales and a second part labelled 10,000 progressions in all the Mozart piano sonatas. He confirms that he has dealt with modulations by labelling triads according to the current key. My Figures 4 and 5 are scanned from his book. There is a significant difference between the corresponding major panels for Bach and Mozart. Though both composers use iii very sparingly, Mozart's much higher frequency of the transition iii-I is remarkable. In contrast, Bach sends iii mainly to IV. I find the relative frequency of the interrupted cadence V - vi surprisingly low with both composers. In the minor mode, Mozart always sends the relative major chord III to iv.

First	Second	I C	♭II D♭	II D	♭III E♭	III E	IV F	♯IV F♯	V G	♭VI A♭	VI A	♭VII ♭B	VII B
I	C		3	14		2	20	1	38	5	8	1	8
♭II	D♭	13		**53**				7	20				7
II	D	5	3		1	4	1	7	**63**	2	8		6
♭III	E♭	10	10						40	40			
III	E	5		11			37		5		37		5
IV	F	40	3	15		6		4	24		1	1	6
♯IV	F♯	19							**81**				
V	G	**85**		4	1	1	2			4	3		1
♭VI	A♭	12	8	31		4	12		15		12	8	
VI	A	9	5	**65**		2	9	5	2				2
♭VII	♭B				**83**				17				
VII	B	**79**				9		3	6	3			
	%	30	2	12	1	2	8	2	24	3	5	1	4

Table 5: Relative frequency of progressions from one triad to the next, after Temperley. Values are percentages such that each row sums to 100%. The largest values are picked out in red. The Roman letters here apply to major, minor and diminished triads.S

My own modest amount of original by-hand analysis has included 695 chord changes in 4 Bach chorales and 20 English hymn tunes, all in major keys, plus a separate count of 1235 chord changes in short pieces[1] in a minor key. The results for relative frequency in major pieces are in Table 7, expressed as percentages such that again each row adds to 100%. The categories are those intended for Program 1, namely those whose root has Roman numeral I, ii, iii, IV, V, vi, vii^o, ♭VII or Ic. In C major these would correspond to triads C, d, e or E, F, G, a, b^o and B♭. A few comments are in order:

- some of the chords are created by voice-leading, sandwiched between two more stable and common chords. An example is the 'passing 6 4' between Ia (root position) and its first inversion, Ib in the sequence Ia - Vc - Ib, the bass moving by step C, D, E.
- the sequence I - IV - I is common, particularly at the beginning of a piece. It serves to extend the tonic harmony. There are a few similar sequences such as V - iic - V and vi - iii - vi where a chord is extended by inserting another whose root is a 5th distant. The move from V to ii has been described by George Pratt as a 'reverse thrust'[2].
- Ic does not always go directly to V; in a few cases its resolution is delayed by moving to IV or ii before then going to V.
- In several places the III chord was major, with a sharpened 3rd. This was being used as the dominant of vi to which it progressed immediately.
- The ♭VII chords appears in only two hymns from the late 16th century when the ecclesiastical modes still had some influence. One had a central section which was essentially in the subdominant harmony, where B♭ is IV of IV.

[1] In a minor key were 15 English hymn tunes, 6 Bach chorales, the Bach a minor two-part invention, two movements from a cello sonata by B. Marcello, and short pieces or extracts by Handel, Eccles, Haydn, Heller, Beethoven, Schubert and Schumann.

[2] The Dynamics of Harmony: Principles and Practice, OUP, 1984. §14.11, p 128.

MAJOR %		I	ii	iii	IV	V	vi	vii°
		0	2+6	4	5	7	9	11
I	0		16	2	22	42	9	10
ii	2+6	8		4	1	74	8	6
iii	4	5	11		37	5	37	5
IV	5	42	20	6		25	2	6
V	7	88	4	1	2		3	1
vi	9	10	73	2	10	2		2
vii°	11	81	3	9		6		

MINOR %		I	ii°	♭III	iv	V	♭VI	♭VII	vii°
		0	1+2+6	3	5	7	8	10	11
I	0		20		22	41	5	1	9
ii°	1+2+6	10		1	1	79	2		7
♭III	3	10	10			40	40		
iv	5	43	24			25		2	6
V	7	88	4	1	2		4		1
♭VII	8	14	45		14	18		9	
♭VII	10			83		17			
vii°	11	87	3			7	3		

Table 6: Data extracted from Temperley's tables to separate frequencies of consecutive triad pairs in major (top panel) and minor keys (lower panel). The second columns and second rows in the header give the root distance from the tonic in semitones. Table entries are in percentages with each row adding to 100%. Zero values are left blank.

a) Mozart: major keys

%	I	vi	IV	ii	vii°	V	iii
I	*	5	15	13	5	62	0
vi	9	*	14	52	4	21	0
IV	50	0	*	19	10	21	0
ii	1	1	1	*	18	77	0
vii°	82	0	1	0	*	16	1
V	94	4	1	0	1	*	0
iii	67	33	0	0	0	0	*

b) Mozart: minor keys

%	i	VI	iv	ii°	vii°	V	III
i	*	5	8	9	11	67	0
VI	3	*	19	58	13	6	0
iv	43	0	*	10	9	39	0
ii°	2	0	0	*	27	71	0
vii°	74	0	1	1	*	25	0
V	81	8	5	0	5	*	0
III	0	0	100	0	0	0	*

c) Bach: major keys

%	I	vi	IV	ii	vii°	V	iii
I	*	9	28	15	6	41	1
vi	12	*	11	30	9	33	5
IV	22	2	*	13	23	39	0
ii	1	1	0	*	25	71	0
vii°	91	3	2	0	*	4	1
V	82	9	7	1	0	*	0
iii	3	32	52	3	3	6	*

d) Bach: minor keys

%	i	VI	iv	ii°	vii°	V	III
i	*	9	20	18	12	41	1
VI	3	*	14	54	8	19	3
iv	22	0	*	14	15	48	0
ii°	1	0	0	*	7	89	3
vii°	81	0	3	0	*	15	1
V	80	10	6	0	2	*	2
III	6	31	25	6	13	19	*

Figure 4: Extract from 'A Geometry of Music' by D. Tymoczko. Probability in % that the given row-labelled triad will move to the column-labelled chord. Percentages in Mozart's and Bach's major and minor key passages.

	MAJOR %	C	d	e/E	F	G	a	b^o	B♭	
		I	ii	III	IV	V	vi	vii^o	♭VII	Ic
C	I		9	2	33	39	11	4	1	1
d	ii	8		14	6	58	6	2	2	6
e/E	III	3	3		20	13	63			
F	IV	29	7	3		36	3	6	4	11
G	V	68	5	5	4		16		1	1
a	vi	5	37	14	11	25		4		4
b^o	vii^o	67		17		6	6			6
B♭	♭VII	14			57	14	14			
	Ic		5		14	81				

Table 7: Frequencies of chord transitions, row to column, obtained by the author from 4 Bach chorales and 20 English hymn tunes all in major keys, transposed to C as reference tonic. Values are in percent such that each row sums to 100%.

- Bach's harmonisations are usually more adventurous than English hymn tunes. For example, chorale 26 has a b minor chord in a chorale nominally in F major.

The minor mode is more complex because it has both ascending and descending variants, furnishing extra notes[3]. Moreover, pieces which start in the minor have a tendency to migrate to the relative major. My data for short pieces in minor keys is tabulated in Table 9 and described in the next section. I have respected the complexity and used more categories of chord than for pieces in the major. The table is divided into nine blocks by three categories of row and the same three of column. The first category is major triads, the second minor triads, whilst the first three rows/columns of the third category are 6 4 second inversion triads, and the second set of three are diminished chords, including diminished 7ths, built from minor thirds. I have not attempted a close comparison with Tymoczko's values for Bach chorales and Mozart piano sonatas, given in Figure 4. My data gives the most common progressions for pieces in c minor as

G to c, c to G, B♭ to E♭, followed in frequency by E♭ to B♭, B♭ to c c to f and c to B♭.

The first two of these are consistent with Tymoczko's values. I have not separated the diminished triads ii^o (D+F+A♭) and vii^o (B+D+F) since they overlap and are each incomplete diminished 7ths. In many pieces diminished chords are represented by only 3 or their 4 notes. Some of the chord progressions which occur infrequently are associated with strongly directed bass lines, such as scales moving with chromatic notes.

4 Program 1 to generate chord sequences

Program 1 will generate a sequence of triads, of any specified length, notated by Roman numerals I, II or ii , ... vii^o, Ic, etc. , which are in a plausible musical order. Layouts of the notes within a chord are not specified, nor is there any melody or rhythm. Here is an example of the output in its barest form, listing 25 triads in a major key:

I V I V I IV V I IV ii V I V I IV ii V vi ii V I vii^o Ic V I

[3] Indeed, if the supertonic is also flattened, the descending minor scale reverts to the phrygian mode.

Number of chords of each type:
I = 8, ii = 3, iii = 0, IV = 3, V = 8, vi = 1, vii^o = 1, Ic = 1.

As is common in music texts, upper case denotes a major triad and lower case a minor one. ^{o} denotes a diminished one. Ic denotes a second inversion 6 4 chord which usually resolves to V. I debated with myself whether Ic should be included as a distinct triad since it is essentially V with two notes (C and E in C major) suspended from the previous chord by melodic voice leading. I have included it because it is common and characteristic of the period. The list is not guaranteed to end on any other particular triad; instead the user must truncate it at a suitable position. The reader might think that the list above looks like a student exercise to build a chorale or short piano piece using the given chord sequence.

The program works as follows. It starts with the tonic triad, I or i, and selects subsequent triads in a probabilistic way using a uniform random number generator and a look-up table of relative frequencies of occurrence of the next triad. This next triad is selected by considering only the current one. Program 1 uses different transition probability values for major and minor modes. Such one-neighbour dependence is known as the Markov property of the sequence, so the program generates a Markov chain of triads by Roman labels. The challenge in developing this has been to quantify the probabilities with which one chord will follow another. There are different look-up frequency tables for major and minor modes.

4.1 Major mode

I took my data in Table 7 together with Temperley's in the major panel of Table 6, plus the data on Bach and Mozart in Figure 4, to judge the values to use within the program. The transition matrix actually used for major key chord sequences is given in Table 8. As far as I understand, all the values are consistent with music theory.

MAJOR (%)	I	ii	iii	IV	V	vi	vii^o	♭VII	Ic
I		13	2	28	41	9	6	1	
ii	4		8	3	64	5	10	1	5
iii	5	6		30	10	44	5		
IV	35	14	2		28	2	9	3	7
V	80	3	3	3		9	1		1
vi	8	48	8	10	19		4		3
vii^o	75		8	1	7	4			5
♭VII	14			58	14	14			
Ic		5		12	83				

Table 8: Table of transitions frequencies in % used in Program 1 to generate a sequence of triads in a major key. The first triad of the pair is labelled by row, its successor by column.

Performance can be checked by determining, through experimental runs, the relative frequencies of the various triad types and comparing them with the values found in actual music by the various researchers, as listed in Table 4. I therefore generated sequences totalling nearly 18,000 chords and found the frequencies listed below, in %. They compare quite well with those in the 'major' columns of Table 4, and so give credence to the program.

I	V	IV	ii	vi	vii^o	iii	Ic	♭VII
$31 \cdot 1$	$27 \cdot 1$	$11 \cdot 6$	$10 \cdot 7$	$7 \cdot 8$	$4 \cdot 9$	$3 \cdot 4$	$2 \cdot 5$	$0 \cdot 8$

3-chord sequences such as I - IV - I are common so are not ruled out, but I - IV - I - IV is not common, nor is ♭VII - IV - ♭VII - IV. The program has checks which remove unlikely sequences of 4 triads. However, I - V - I - V - I - V etc. is not uncommon in simpler pieces so is not ruled out. Amongst the nearly 18,000 chords are 6,000 runs of 3 chords. Of these 32% are I - V - I, 8% are I - ii - V, and 9% are I - IV - I. On page 230 of his book Tymoczko lists the most common 3- and 4-chord sequences in his corpus of Bach chorales and Mozart piano sonatas. His numbers are in the following ratios of occurrences, which I compare with those of the program:

$$\frac{I - V - I}{I - ii - (Ic) - V - I} \quad \textit{in Bach} \quad 3 \cdot 0, \qquad \textit{in Mozart} \quad 3 \cdot 6, \qquad \textit{in program} \quad 3 \cdot 9$$

$$\frac{I - V - I}{I - IV - I} \quad \textit{in Bach} \quad 4 \cdot 1, \qquad \textit{in Mozart} \quad 8 \cdot 4, \qquad \textit{in program} \quad 3 \cdot 7.$$

Clearly composers differ in their styles and tastes for various chords progressions, but again I find this satisfactory agreement, and overall judge that the program fulfils its limited function.

As it stands, the results from this program are far from being music. However, they can form the basis for anyone to concoct, by hand, the sort of tedious scale and arpeggio exercise set in tutor books for training violinists and other musicians. Figure 5 gives is an example which elaborates the following program-generated sequence:

I V I ii V vi ii V I IV V I vii^o I V I iii vi ii Ic V I

For an intermediate violin student (about ABRSM Grade 4 or 5) this would be a fair test of tone and intonation.

Figure 5: Study-type violin or clarinet piece invented by the author by elaborating the chord sequence above using one bar per chord, to be played *allegretto*.

			C	D♭	D	E♭	E	F	F♯	G	A♭	A	B♭	B	c	c♯	d	e♭	e	f	f♯	g	a♭	a	b♭♭	b							
			I		II	♭III	III	IV		V		VI	♭VI	VII	i		ii	♭iii	iii	iv	♯iv	v	♭vi	vi	♭vii	vii	Ic	Vc	♭IIIc	dim1	dim2	dim3	
			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	
C	I	1						6							1		1			12					2							1	23
D♭		2																												1		2	3
D	II	3						2		12					1							16					2				2		35
E♭	♭III	4	3		1			7		3	26		47		24		2			11		8			1		2	1		2	1	3	142
E	III	5																															0
F	IV	6			4	1				3			29		3					1		6			2		2						51
F♯		7																															0
G	V	8	8		2	7		2			10		2		**125**					6		3								1	1	2	169
A♭	♭VI	9			2	8		3		1			12		1					13		2					5		5	1		5	58
A	VI	10			2																												2
B♭	♭VII	11	1			**86**		5		5	2	2			41		2	1		1		15							1		1	2	165
B	VII	12																															0
c	i	13	2	1	4	16		20		**74**	11		36				7			43		20					3	9	2	5		15	268
c♯		14																															0
d	ii	15			1	1		1		5			2		5							1					1					2	19
e♭	♭iii	16											1																				1
e	iii	17																															0
f	iv	18	1	2	3	8		2		26	3		19		13		3					1			4		10		1	2		3	101
f♯	♯iv	19																															0
g	v	20	9		8	11		2			5		6		24		1			3				2							2	2	75
a♭	♭vi	21																															0
a	vi	22								1																					1	1	3
b♭	♭vii	23	4			2		1												1										1			9
b	vii	24																															0
	Ic	25			1					24			2				1			3													31
	Vc	26			1										8		1																10
	♭IIIc	27				1							5							2												1	9
	dim1	28			2					4	1		1									3		1			2						14
	dim2	29	2		4															1							1						8
	dim3	30				1				10			3		15		1			4		1					3			1			39

Table 9: 1235 pair-wise chord progressions in small pieces in a minor key, referred to c minor as reference, obtained by the author. First chord in row, second in column. The upper left 4 blocks refer to triads which are strictly major (capitals) or minor (lower case). Thus III would be E major, iii e minor, and ♯iv be f♯ minor referred to c as tonic. Ic is the 2nd inversion of I, a 6 4 triad. ♭IIIc is the relative minor 6 4 chord. dim1 is a diminished triad or 7th which contains the tonic, dim2 contains degree 2 of the scale, and dim3 is the one containing the leading note. Colours pick out the salient values. 8 rows and columns have no non-zero entries. The rightmost column gives the row totals.

			C	**D♭**	**D**	**E♭**	**F**	**G**	**A♭**	**A**	**B♭**	**c**	**d**	**e♭**	**f**	**g**	**a**	**b♭**						
			I	**♭II**	**II**	**♭III**	**IV**	**V**	**♭VI**	**VI**	**♭VII**	**i**	**ii**	**♭iii**	**iv**	**v**	**vi**	**♭vii**	**Ic**	**Vc**	**♭IIIc**	**dim1**	**dim2**	**dim3**
			1	2	3	4	6	8	9	10	11	13	15	16	18	20	22	23	25	26	27	28	29	30
C	**I**	1					26					4	4		52			9						4
D♭	**♭II**	2																				33		67
D	**II**	3					6	34				3				46			6				6	
E♭	**♭III**	4	2		1		5	2	18		33	17	1		8	6		1	1	1		1	1	2
F	**IV**	6			8	2		6			57	6			2	12		4	4					
G	**V**	8	5		1	4	1		6		1	74			4	2						1	1	1
A♭	**♭VI**	9			3	14	5	2			21	2			22	3			9		9	2		9
A	**VI**	10			100																			
B♭	**♭VII**	11	1			52	3	3	1	1		25	1	1	1	9					1		1	1
c	**i**	13	1	0	1	6	7	28	4		13		3		16	7			1	3	1	2		6
d	**ii**	15			5	5	5	26			11	26				5			5					11
e♭	**♭iii**	16									100													
f	**iv**	18	1	2	3	8	2	26	3		19	13	3			1		4	10		1	2		3
g	**v**	20	12		11	15	3		7		8	32	1		4		3						3	3
a	**vi**	22						33															33	33
b♭	**♭vii**	23	44			22	11								11							11		
	Ic	25			3			77			6		3		10									
	Vc	26			10							80	10											
	♭IIIc	27				11					56				22									11
	dim1	28			14			29	7		7					21	7		14					
	dim2	29	25		50										13				13					
	dim3	30				3		26			8	38	3		10	3			8			3		

Table 10: Minor mode transition matrix for chord progressions used in Program 1. Values are derived from Table 9 and expressed in percentages (rounded to nearest integer) such that each row sums to 100%. Row gives first note, column the second. Notes are referred to c as tonic. dim1 contains notes A, C, E♭ and possibly also F♯. dim2 contains D♭, E, G and possibly also B♭. dim3 contains B, D, F and possibly also A♭ so it includes both ii^o and vii^o.

4.2 The minor mode

Because the minor mode has the extra notes brought by the ascending and descending variants, and because pieces which start in the minor have a tendency to migrate quickly to the relative major, I decided simply to use the categories and data values I had collected in Table 9 for the program. The matrix used in the program is given in Table 10. This is obtained from Table 9 by deleting the eight empty rows and columns and normalising the values so that the sum of each row is 100%. I emphasise that this table applies to the totality of short pieces nominally in a minor key, and not just to sections before the music has modulated away from the minor to the relative major or related key. I have not struggled to relate Table 9 to Tymoczko's data on Bach and Mozart in Figure 4 beyond observing that i - V and V - i are the most common chord pairs.

A note is required about the diminished chords labelled dim1, dim 2, dim3. If all four distinct notes are included, these are the only diminished chords, apart from the spelling of the note names. dim1 contains contains the tonic note, dim2 contains the dominant, and dim3 contains the leading note. So in the key of C these are (C, E♭, F♯, A), (D♭, E, G, B♭), and (D, F, A♭, B) respectively. Triad vii^o in the major mode is therefore dim3 with the A♭ omitted.

I ran the Program 1 using the frequencies in Table 10 for 25,000 chords and recorded their relative occurrence as a way of checking that the program reproduces the frequencies reported in Table 9. The most common triads for a piece nominally in c minor are c (21%), G (13%), B♭ (12%), E♭ (12%), f (8%) and g (6%). Surprisingly 57% of all chords are major, only 38% minor. Of these major chords, 4% are 6 4 chords. The diminished chords make up the remaining 5%.

The test, of course, is whether the chord sequences generated from Table 10 sound like musical progressions. As one test example, here is a sequence of 60 chords generated consecutively in one arbitrary run of the program.

i IIIc ♭VII i ii V I IV Ic V i Vc i iv ♭III iv ** i V i V i iv i ♭VII v dim2 II v ♭III v dim3 i IV ♭VII ♭III i ♭VII ♭III v i ♭III iv V I IV V I IV V i iv V i V i ♭VII i V i Vc ii .

Do they make musical sense? The user needs to judge which sections of this are potentially useful. Here I find that the IIIc, ♭VII and I chords so early on do not allow i to be established as tonic. I am therefore inclined to reject the first 16 chords, up to the ** mark, and start with the following i - V - i. The rest of the sequence can continue, though the last two chords are deleted so it can close with a perfect cadence. Here (Figure 6) are the remaining 42 chords in musical notation, again with fairly arbitrary voicing. I emphasise that the layout of these chords and their notation as music were not produced by Program 1, but by me merely to illustrate how they might sound.

Playing through Figure 6, it does sound like the skeleton of a piece in a minor key and I find no jarring changes. There are a few surprises, however, particularly the two diminished chords (which I have filled out into 7th chords) and the C major section in the middle. I have therefore looked at my musical sources to find cases of these less common progressions. The database has only one example of the v - dim2 - II progression in bar 3 above; it appears at the beginning of the second section of the short Siciliano in a minor, number 11 in Schumann's Album for the Young, Op 68. The relevant few bars are given in Figure 7.

The diminished chord is not a complete 7th, but Schumann's harmony is not greatly altered by adding the missing G, the implication being that it resolves to an F♯ missing from the B major

Figure 6: Block chords chosen by the author to represent the chord sequence above, after **.

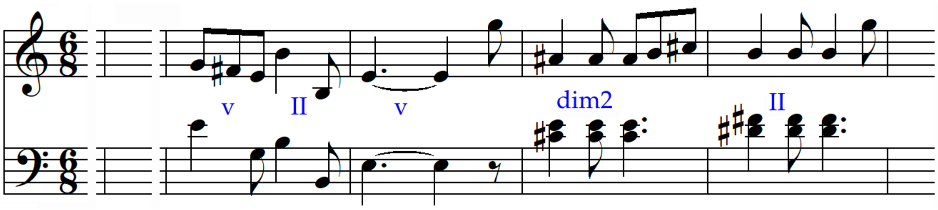

Figure 7: A few bars from Schumann's Siciliano, Op 68 No 11.

II chord. Regarding the v - dim3 - i progression in bar 4, I could not find a convincing example of this, though there are instances of V - dim3 - i, v - dim3 - iv, i - dim3 - v, i - dim3 - i and i - dim3 - iv. Clearly v - dim3 - i is rare but not musically impossible. Let us therefore accept the transition matrix for minor, Table 10. In addition to the Roman notation, the output from Program 1 is also in the notation of Tables 8 and 10, which numbers triads from 1 to 30. Subsequent programs which take input from Program 1 read it in this Table 8 format.

4.3 Principles of modulation

The sense of departure and return in music is emphasised if the tonal centre moves and is later restored. I explained in §3 that a distinction is best made between short term colouring of the harmony and the transition to an extended sequence of chords that is clearly centred in another key. The frequency probabilities used in Program 1, tabulated in Tables 9 for major and 10 for minor (§4.2), were obtained on the assumption that they apply within only one key. In the minor mode, as Figure 6 illustrates, chords indicating the relative major occur automatically within any extended sequence generated by the program. Taking C as the reference key, in both major and minor modes a subsequence of triads can be given the temporary colouring of the dominant, G major, by sharpening F to F♯[4].

Textbooks point out that there is a logic to the way music moves to different keys. It most naturally follows much the same progressions as do the triads themselves. The most common are:

[4] The exception is that the F of the diminished chord vii° cannot be sharpened when the next chord is C major, since b minor to C major is a very unlikely progression. The F should fall to E of the C triad.

- tonic key to dominant (I → V) followed eventually by
- its return, often via the subdominant, IV → I.
- to the relative minor and back: $(\text{I} \rightarrow \text{vi})_I$, $(\text{i} \rightarrow \text{III})_{vi}$.

Large swerves into distant, unrelated keys – popular with some late romantic composers – require chromatic alteration of chords, enharmonic tricks and other devices and do not always produce a convincing musical effect. Hence, for the time being, we concentrate on diatonic modulations to closely related keys.

C	I	**vi**	II	V	G			**I**	IV
C to G	C	**a**	D	G	**G to C**			**G**	C
G:	IV	**ii**	V	I	C			**V**	I
C			**I**	IV	F	I	**vi**	II	V
C to F			**C**	F	**F to C**	F	**d**	G	C
F			**V**	I	C	IV	**ii**	V	I
C	I	**ii**	III	vi	a	i	**iv**	♭VII	♭III
C to a	C	**d**	E	a	**a to C**	a	**d**	G	C
a	III	**iv**	V	i	C	vi	**ii**	V	I
c	i	$\mathbf{i7^{o}}$	II	v/V	g	i	**v**	I	iv
c to g or G	c	$\mathbf{c7^{o}}$	D7	g/G	**g to c**	g	**d**	G	c
g	iv	$\mathbf{iv7^{o}}$	V	i/I	c	v	**ii**	V	i

Table 11: Modulation schemes. Upper three panels: modulation in major from tonic to dominant, tonic to subdominant, and tonic to relative minor, and back. Lowest panel : modulation in a minor key tonic to dominant and back. Pivot chords are in bold. Chords in the starting key are in the row above the note names, and those in the new key below.

The type of diatonic modulation considered here requires three triads in succession: i) a ‘pivot’ chord, ii) the dominant or, stronger, the V7th chord of the new key, and iii) the new tonic. The pivot chord naturally exists in both the old and new keys; it is the ‘doorway’ through to the new key, equally recognisable from either side. The dominant requires the 3rd of the triad to be raised by a semitone to form V in the new key, which then resolves to I_{new}. Table 11 sets out several schemes for modulating. The top panel is from tonic to dominant in a major key (left) and back (right), using C as the reference tonic. Observe the circle of 5ths relation of vi - ii/II - V. The second panel is from tonic to subdominant in a major key (left) and back (right), the chord sequences being swapped over from the panel above. The easiest modulation might seem to be to the subdominant since the current tonic triad is itself V of the new key. Forte, however, comments that this is the most problematic diatonic modulation because IV immediately assumes the primal role, so we hear the piece as being in F rather than in C. The main role of this modulation is to return from G major to C major.

The third panel in Table 11 shows transition to the relative minor from a major key. This requires the G of the iii triad in C major to be raised to G♯, the leading note of a minor. Again the new key is felt most positivity if the note D is retained from d minor as the 7th of E major.

Transition back follows the descending melodic minor scale with no sharpened notes, and appears as the classic dominant cadence formula vi - ii - V - I. With pieces nominally in the minor mode modulation to the relative major, a to C or c to E♭, is common. It is the only modulation which does not require chromatic alteration of any triad. One thing to note in Table 10, which summarises my data, is that in a third of instances, i moves directly to ♭VII without an intervening iv.

The lowest panel shows one scheme for modulating from a minor tonic to its minor or major dominant, and return. The diminished 7th chord on C contains (C, E♭, F♯, A) and readily slips into the dominant 7th (C, D, F♯, A). In the return, composers avoid the immediate juxtaposition of g and G (minor:major), usually separating them with ii or another chord.

In practice modulation is best dealt with after the phrase and cadence structure of the piece has been decided, and is therefore reserved until Program 6, §8.1.

5 The simple melodic line

We now turn from the harmonic progression of triads to describe Programs 2 and 4 which can generate a single melodic line of even notes according to the supposed 'laws' of melody and counterpoint. Appendix 1 describes Program 3 which converts the output to a Lilypond file for score engraving.

5.1 Anatomy of a melodic line

When I wrote the first version of the program in the 1980s, my guide was the antique textbook 'Counterpoint for Beginners' by C. H. Kitson (OUP 1927). Kitson teaches 'species counterpoint', a scheme developed during the 17th century and codified by Johann Fux in his classic thesis *Gradus ad Parnassum* written in Latin and published in 1725. The teaching method assumes that a single melodic line called the *canto fermo* (fixed voice, CF) is first created in even notes. Strict rules govern the CF, derived from the plainsong singing practices in churches and monasteries. The basic 'first species' adds a second voice line to this, note against note, according to strict rules of combination. Second species sets the added voice moving at twice the rate of the CF, so producing two notes against each one of the CF, most of which are passing notes. In third species there are four notes in the counterpoint to every one in the CF. Fourth species is concerned with suspensions, while fifth species, also called floral counterpoint, consists essentially of a combination of the earlier species, but still with only one voice added to the CF. The last chapters of Kitson's monograph deal with two parts in fifth species against the canto fermo, by which time the student can write 'academic' polyphonic music in a style resembling that used in church singing up to the early 18th century. Despite the didactic nature of this species scheme, it remains a fair starting position for the design of melodic voice lines which fit together. The results fall short of being music because of the lack of a phrase structure punctuated by cadences and other rhythmic interest. (That will be addressed in Program 6.) Kitson emphasises the harmonic basis of counterpoint from the 17th century onwards so his teaching on 2- and 3-part writing is essentially harmonic progression through the smooth moving of independent voices.

I supplemented Kitson's rules and guidance by counting the frequency of occurrence of the intervals between every pair of adjacent notes in about 25 pieces of music. This was done by entering the soprano and the bass lines of several pieces – hymn tunes, psalm chants, slow movements from concerti grossi by Corelli and Handel and some popular songs – into a program which calculated the horizontal pitch difference between successive notes in the soprano and, separately, in the bass. It thereby created a table covering intervals from −12 to +12 semitones. I chose pieces which move in chords with easily identifiable soprano and bass notes, and recorded mainly the structural notes which occupy a whole beat, so omitting all but a few passing notes. In all there were 982 intervals in the soprano and 841 in the bass. The results are plotted as percentage frequencies in Figure 8.

About 10% of notes are repeated. In hymns this often occurs across a cadence at the end of a line. A step of 1 tone is by far the most common, especially in the soprano (orange), where a step down is more common than a step up – a surprising finding. Steps by either a major or minor 3rd make up about 18% of intervals in both soprano and bass, with downward moves more common than upward. The diminished 5th (6 semitones) is very rare (< 1%). There are some significant differences between the upper (orange) and bass (dark blue) voices:

- stepwise movement is more common in the upper voice. In the soprano 54% of intervals are either 1 or 2 semitones, compared with only 40% in the bass,
- the bass moves up by a perfect 4th (5 semitones) and down by a perfect 5th (7 semitones), and up or down by an octave, much more frequently than does the upper voice.
- a leap of a minor 6th (8 semitones) is more likely to occur in the upper voice and be in the upwards direction,
- though a leap by 7th is rare, it hardly ever occurs in the bass, only in the soprano.

5.2 Program 2: generate elementary vocal lines

The first stage in this scheme is to generate a valid *canto fermo* (CF) in one voice, and Program 2 does this. The CF moves mainly by step; its leaps are few and rarely greater than a major 3rd. This makes it easy to sing, even if it is rhythmically sterile. C is taken to be the reference tonic throughout this article. Kitson gives these rules:

1. the CF starts on C or G,
2. the CF ends with last two notes D then C, supertonic falling by step to tonic,

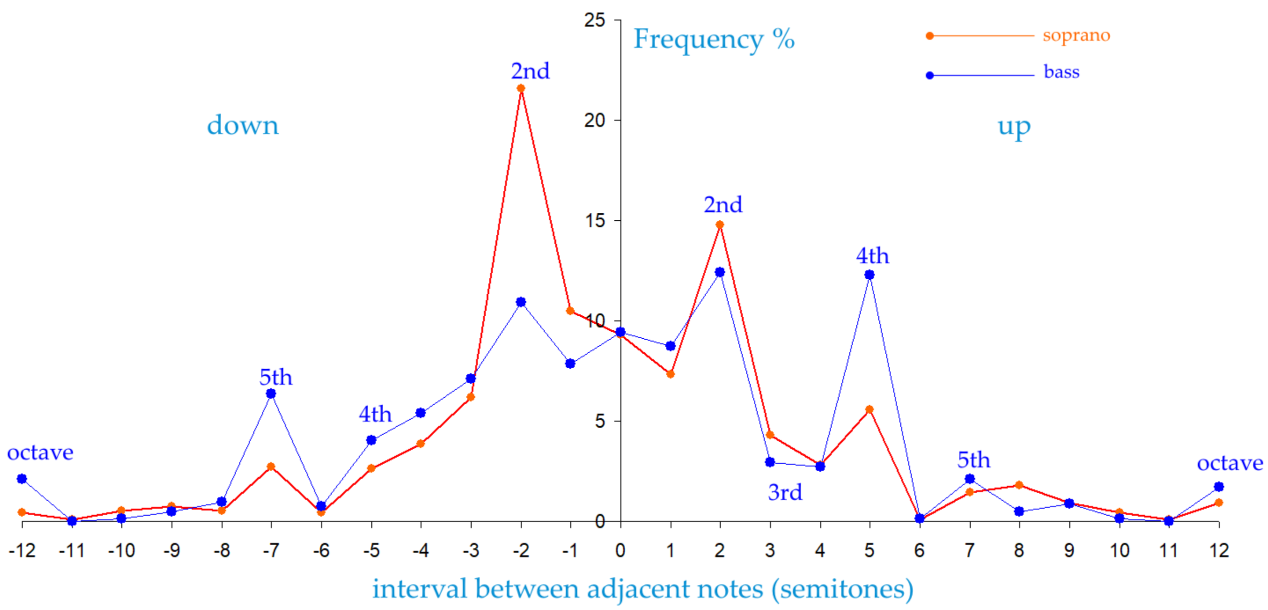

Figure 8: Observed frequencies of horizontal (melodic) intervals between adjacent notes, counted in semitone steps. Negative values denote a step or leap downwards, positive upwards.

3. his examples imply that no note is immediately repeated; that would just be a change in rhythm,
4. the interval between two adjacent notes can only be
    - an octave, provided the preceding and succeeding notes lie inside that octave,
    - a minor 6th, provided the preceding and succeeding notes lie inside that 6th,
    - a perfect 5th,
    - a perfect 4th,
    - a major or minor 3rd,
    - a major or minor 2nd,

   all provided the notes are chosen from the major scale, or ascending and descending forms of the minor scale. Thus chromatic notes, leaps of a major 7th or major 6th, and leaps of any augmented or diminished interval are all forbidden. The origin of these prohibitions was probably that these intervals are difficult to sing in tune.
5. three successive leaps in the same direction are not allowed,
6. three successive notes cannot span a major 7th between their 1st and 3rd notes,
7. three successive notes can span a minor 7th between their 1st and 3rd notes provided both intervals are not 4ths, but only if the preceding and following notes both lie inside that minor 7th.
8. the melody is intended to be singable and so remain within a vocal range of about one octave plus one perfect 5th from highest note to lowest.

Informed by Figure 8, and bearing in mind that some tunes might be played on instruments rather than sung, I have introduced some modest additions and changes from this, as follows:

1. the soprano can start on C, E or G, but the bass starts on C. Both voices close on a C.
2. the range can extend to over 2 octaves, in which there may be a shift of an octave; that is relocating the pitch centre an octave higher or lower.
3. in the upper voice a note can be repeated once, and in the bass repeated twice, i.e. sounded three times,
4. both a major 6th and a diminished 5th are allowed provided the notes either side lie within the interval,
5. in the minor key the ascending and descending forms of the scales should be observed; thus C B♭ A♭ G A B C,
6. as an alternative cadence in the soprano, B♮ can rise to C, provided that when another voice is added[5] it falls D to C.
7. the bass can close G to C, and indeed this is probably better than B C or D C,

[5] Kitson states that the counterpoint voice should always close B to C whilst the CF closes D to C.

8. Macpherson[6] recommends that the leading note, B♮ should not fall unless as part of a descending scale C B A. Having looked at some pieces I find that this is reasonable advice in the minor, but over restrictive in major.

9. a descending augmented 4th or 5th is not allowed between next neighbouring notes, e.g. B down to F in the major, or A G E♭ in the minor, or A F E♭.

Program 2 creates a soprano or bass line according to the above rules with my modifications. It does so by using a random number generator to select the magnitudes of the intervals (not the actual notes) according to a look-up table of likelihood derived from Figure 8 and described below. The candidate next note is checked against all the rules for C major or minor and accepted only if valid. The first check is on the candidate new note itself to confirm it lies within the scale. Then it is checked with the current note to prevent unvocal intervals. A third check examines the run of three notes to sieve out wide intervals between next-but-one notes, three repeated notes, successive 4ths in the same direction, the leading note being an upper auxiliary note, and other undesirables. Yet a further check is applied to a run of four notes to catch three leaps in the same direction, etc. When a user-specified number of verified notes have been generated, the program will continue until the next D - C pair or B - C pair is created, at which point it terminates. The program creates a text file with the sequence of notes recorded by number of semitones in MIDI notation when middle C = 60 and ±1 means a change of one semitone.

One of the subtleties of writing the code has been dealing with the ascending and descending forms of the minor scale. Another has been distinguishing ambiguous intervals, such as the diminished 5th, which may be allowed, from the augment 4th, which never is. The minor mode has a chameleon character, frequently morphing into the relative major. The minor quality comes essentially from the flattened 3rd degree of the scale: E to E♭ in the key of C. The leading note B is flattened only when the voice line descends to its next note, and the 6th scale degree, A, is usually flattened too when the next note descends, though the composer has some choice in this. After Program 2 has generated a vocal line in the minor mode, it runs tests on the whole line to enforce the ascending and descending forms of the minor scale. The harmonic form, with an augmented 2nd A♭ to B♮, is not allowed. However, because of ambiguities which can still arise over passages such as C B♭ A♭ (or A) B C, the user can save the line to file both before and after these post-creation tests, and make a personal choice on each A or A♭.

The relative frequencies of intervals clearly depends on the rules limiting successive notes, but can also be changed to a limited extent by weighting the random number used to select candidate next notes. I have distinguished a bass line from an upper voice, and weighted the random selection so that the frequencies of intervals observed in test runs is fairly close to those found in the pieces of music surveyed, in Figure 8. For example, for the soprano voice, both major and minor, the weightings at each interval, in semitones, are

0	1	2	3	4	5	6	7	8	9	10	11	12
45	220	240	95	80	80	20	60	55	40	20	0	45
45	265	505	600	680	760	780	840	895	935	955	955	1000

(0 = same note, 12 = octave.) The cumulative weights are in the bottom row, and the random number generator selects with an integer in the range 1 to 1000.

This random selection gives only the magnitude of the interval; its direction up (+1) or down (−1) is first decided by a second random number. The only control imposed upon the overall

[6] Stewart Macpherson 'Melody and Harmony' page 5. publ. Joseph Williams Ltd. 1920.

Figure 9: Samples of *canto fermo* style vocal lines generated by Program 2, converted by Program 3 and engraved with Lilypond.

shape of the melodic line is by biasing the direction of the new interval depending on the pitch of the present note in order to stop the line climbing or descending indefinitely. This is done by selecting a random number in the range 1 to 200 and setting the threshold value in this range depending on the pitch difference of the present note from a central value; random numbers above this cause a downwards direction. Pitch is measured on the MIDI scale in which Middle C = 60 and ±1 denotes a change by 1 semitone. Thus, for a soprano voice the central value is B♭, MIDI 70, in the centre of the treble clef. The threshold value is given by $INT(100 - 4{\cdot}6(p - 70))$ where p is the pitch on the MIDI scale of the last note. Therefore notes close to this B♭ are about as likely to move upwards as downwards. For the bass line the central value is D on the middle line of the bass clef, and the corresponding formula for the threshold is $INT(100 - 4{\cdot}7(p - 50))$. These formula give thresholds 0 and 200 at MIDI 48 and 92 respectively in the soprano, and at MIDI 9 and 72 in the bass, so limiting the possible range of any vocal line to about $3\frac{1}{2}$ octaves. This scheme imposes only short range control over the direction of a vocal line, and comes significantly into play only when the notes have moved far from the centre of their respective stave. It is not enough to stop meandering around the central B♭ in soprano or D in the bass. To limit the tendency to meander, the program looks at each run of three notes and deletes any immediate repetition of that three.

In Figure 9 are four examples of short *canto fermo* voice lines generated by Program2. They are printed without bar lines since there is neither phrase structure nor beat. Overall I find the created lines to be singable and hence satisfactory, though in themselves few offer any musical interest. Their meandering shape is a major weakness.

5.3 First species counterpoint

A further capability of Program 2 is to fit a bass voice line to an upper voice or *vice versa*, note against note, as with the exercises in Kitson's monograph where the *canto fermo* is given and a counterpoint then created to fit. A soprano or bass line is generated first by one version of the program and provided to a second version as a DATA statement of pitches. The counterpoint voice line is generated by the same algorithm but with additional limitations on the next note at each step. Let the current notes in soprano and bass be s_0 and b_0 respectively. Candidate next notes are proposed in the upper voice, s_1, and bass, b_2, using the random number method above with filtering rules, as if the voices were independent. The additional filters are based to the vertical intervals between the proposed new note pair, $s_1 - b_1$, and at the current position $s_0 - b_0$, and also on intervals $s_0 - b_1$, $s_1 - b_0$. The additional rules for matching soprano to bass include:

- the bass should at no point be higher in pitch than the soprano, but unison is allowed.
- the parts should not overlap on consecutive beats: that is $b_1 < s_0$ and $s_1 > b_0$ for every two pairs of notes s_0, b_0 and s_1, b_1,
- no parallel unisons, octaves or perfect 5ths.
- the same interval between S and B should not persist for more than 3 beats; in this major and minor 3rds are counted as the same interval,
- the allowable intervals[7] are unison, minor and major 3rd, diminished and perfect 5th, minor and major 6th, and all of these plus 1 or 2 octaves.
- a diminished 5th must contract to a smaller interval at the next beat.

To illustrate the type of result, Figure 10 gives examples of a bass fitted to a soprano rising major scale, and of a soprano fitted to a falling minor scale in the bass. A more extended example is in Figure 11. Here the bass voice was generated first. I changed by hand the two notes F and B♭ marked with ×, raising their pitch by exactly one octave as this improved the line. We may be disappointed by the movement in almost equal intervals at A...A. Had the two lines been exactly the same number of semitones apart for more than 3 beats, a rule would have prevented the fourth such interval. These examples, of course, have all been produced without reference to any harmonic basis beyond avoiding discords and maintaining independence of the voices; the harmony is melodically driven.

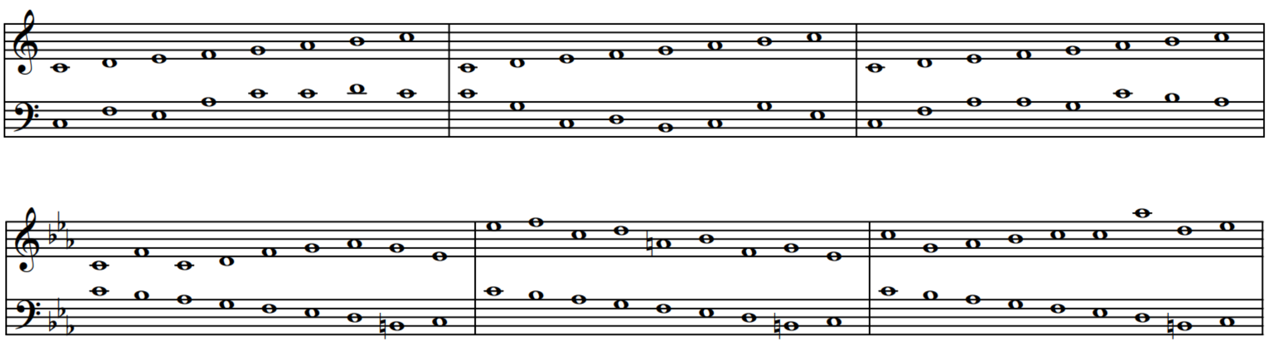

Figure 10: First species counterpoint applied to scales in soprano or bass.

First species counterpoint is not meaningful music, but we might ask whether a rhythm and phrase structure could be added by hand to Figure 11 to turn it into a very short piece. One attempt is given in Figure 12. Whatever pleased the ears of monks in the era of plainsong, our ears today expect an harmonic basis to music, with phrases punctuated with cadences. Hence the division of the notes into two-bar phrases. Hearing this, you can sense the implied harmony. The piece sounds like an ancient carol. The two notes with accidentals in brackets sound better without their pitches being raised. Unlike harmony, the bass of the final cadence is not G to C.

[7] In 2-part writing a perfect 4th is regarded as a discord.

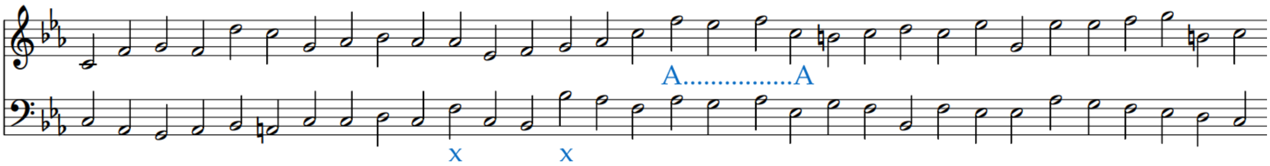

Figure 11: Longer example of 'first species' counterpoint generated by Program 2.

Figure 12: A metrical version of Figure 11.

5.4 Program 4: guide curves to shape the melodic line

The obvious weakness of most of the CFs in Figures 9 and 11, and in many others I have examined, is their indefinite shape in pitch. Too often they meander around or zig-zag. Good musical phrases and longer paragraphs have a definite shape in pitch. In a companion article on *www.mathstudio.co.uk* entitled 'Fitting a curve to 3 or 4 points' I list 14 phrases which exemplify common shapes. Some rise or fall in pitch almost in a straight line. Others are arches, either rising to a top note then falling, or *vice verse*. Yet others rise, fall and rise again like the opening theme of Beethoven's Eroica symphony. In §11 we look in greater detail at the structure of melodic lines. There Figure 47 shows the 14 examples of phrase shape. On a longer timescale the average pitch of successive phrases may rise or fall to give the extended piece a recognisable shape. Interesting music has pitch shape on at least two length scales – phrase and paragraph.

In this subsection I describe a modification to Program 2 to steer the melody towards a well formed curve. The approach dispenses with the local check, $INT(100 - 4{\cdot}6(p - 70))$, on the direction of the next interval described in §5.2 and replaces it with a global control curve spanning to the expected end of the piece. The actual notes are selected from the notes of a sequence of triads previously generated by Program 1. Program 4 therefore differs conceptually from Program 2 in that the single melodic line it produces is governed by a pre-defined harmonic framework, rather than the purely linear melodic progression of Program 2. We might regard Program 2 as producing music which belongs to church singing traditions before 1700, while Program 4 belongs to the musical thinking of the 18th century.

Given a sequence of triads as produced by Program 1, the concept is to define a curve in pitch between start and end, rather like drawing a smooth curved line across the stave of musical notation. Notes are selected from the possible notes in each triad so that they lie close to the guide curve. Specifically, random numbers are again used to select the magnitude of the next interval, but its direction, ascending or descending, is whichever places the next note closer to the guide curve.

Programs 2 and 4 are therefore closely related and serve essentially the same simple purpose of producing a melodic line in notes of the same duration.

There are several ways in which a curve to guide the musical line can be described mathematically – by a function such as a polynomial, a superposition of trigonometric functions, by a Bézier curve, or a cubic spline through given points. The companion article cited above examines the mathematics of candidate curves. The simplest curve is a parabola placed in a symmetric arch over the whole CF, with starting and ending notes at the same pitch, which we can take to be S where middle C = 60 on the MIDI scale of semitones. Suppose there are $n+1$ notes in all up to the closing cadence, indexed from 0 to n, and that the central highest point of the parabola is at pitch P above or below S. The formula for the target pitch of the kth note along this parabola is

$$P(k) \;=\; S \pm P \mp \frac{4P}{n^2}\left(k - \frac{n}{2}\right)^2, \qquad 0 \le k \le n. \tag{1}$$

where the upper sign is taken for a rising arch and the lower for a curve that dips to a trough before rising to the final note. We have to estimate n because the user only specifies the minimum number of notes, after which the program continues automatically until a closing cadence turns up. I have taken this to be typically an extra 6 notes. Bear in mind that the random selection of intervals means that this parabola is only a template to guide the shape. The notes are otherwise produced as before. The three examples in Figure 13 were produced in this way, with a minimum of 16 notes specified and $P = 14$ in the upper and lower cases, and -14 in the second CF (in the bass clef). On the upper CF the guiding parabola is marked with red dots. The second CF wanders around too much in bars 3 and 4, and the bottom CF, in c minor, has more large leaps than the other two, but both still roughly have the required melodic curve. We might agree that these vocal lines are more satisfactory than those in Figure 9, which lacked a global constraint.

Figure 13: *Canto fermo* produced by computer guided by a parabola according to Eq. 1 with $n = 16 + 6 = 22$, $P = \pm 14$.

Clearly more varied melodic curves could be generated by defining guide curves which undulate. For example the form $\sin k - 1{\cdot}5 \sin 2k,\; 0 \le k \le \pi$ dips initially then rises to a higher peak before falling back to its starting value. This can be scaled to fit $n+1$ evenly spaced notes:

$$P(k) \;=\; S + P\left(\sin\left(\frac{k\pi}{n}\right) - 1{\cdot}5\sin\left(\frac{2k\pi}{n}\right)\right), \qquad 0 \le k \le n. \tag{2}$$

Five examples are given in Figure 14. The top two have P set to 8 semitones and the melodic line has closed an octave higher than intended. In the third line the intended shape is obtained and the CF closes at the same pitch as it started, but the range – almost two octaves – could be considered too large. Reducing P to 6 or even as low as 4 and requiring a minimum of 24 notes reduces the expected rate of change of the guiding curve and allows the program more readily to find valid notes which tend to follow the guide. The bottom two CFs are examples. I find that if P is too large and n, the number of notes, too small, the program failed to find a valid melodic line which satisfied all the rules. Typically it would stall part way through creating the sequence, unable to find a valid next note. A brief comment on the bottom CF in Figure 14, in c minor: the minor key is more ambiguous and flexible than the major in the notes it allows. We would probably wish to change the B♮ at the beginning of bar 2 to B♭ though the B♮ is not unmusical. I have not sought to apply rules too rigorously in the minor.

Figure 14: Five computer generated *canto fermo* guided by the sine curve formula of Eq 2.

The illustrations in Figures 13 and 14 show that, within limits, it is possible to steer the overall shape of a melodic line. Our musical judgement needs to be used, of course, not to force the voice line to follow the guide curve so closely that it moves almost entirely in small steps. The line must have enough freedom to incorporate a few leaps larger than 4 semitones, and so attain the statistical frequencies recorded in Figure 8. By so doing melodic interest can be maintained so that the line is not so stepwise as to be too bland, nor so full of leaps as to be an unmusical jumble. A few modest leaps are enough to add vigour to an otherwise stepwise melodic line. The challenge is to have the 'best of both worlds'; local shape of phrases within a global overall arching structure. The later programs in this suite add phrase structure, rhythm and decoration to the bare skeleton output from Programs 2 and 4, 5, and 7. It is necessary that *each phrase* have a satisfactory shape. It requires several notes to trace out an arc, so phrases are apparent mainly in the surface decoration of the music, much less in the underlying harmony. We therefore see that guide curves need to be on two length scales – a global scale across the whole piece (or section) and the local scale of each phrase.

In §11 we return to the creation of acceptable melodic lines and discuss what makes a good tune.

6 Program 5: towards four-part harmony

In its basic form four-part harmony requires four vocal lines to be 'drawn' across the stave, note against note, thereby creating chords through their vertical alignment. Recall that Program 2 creates a single vocal line using only local note-to-note control, while Program 4 creates a single line using the global control of a guide curve. The logical extension of Program 2 is to continue thinking primarily in terms of melodic lines and add a similar second, third and even fourth voice in counterpoint, note against note, according to further rules of allowable vertical (harmonic) intervals as set out by Kitson and other teachers. Early in his book, however, Kitson points out the *harmonic* implications of a single vocal line, and regards two-part writing as devising a sequence of incomplete triads in a logical *harmonic* progression. This points to developing Program 4 rather than Program 2. Four-part harmony in the styles of the 18th century is best developed from a harmonic basis, namely the sequences of triads generated by Program 1. Two, three or four voice line guide curves should be laid across the harmonic progression so that the notation looks like a woven fabric of vertical chords with melodic lines stretched horizontally through them. Four such lines could correspond to the SATB voices of a choir and the result would resemble a harmonised hymn tune.

6.1 The Program 5 algorithm

The basis of Program 5, therefore, is to 'draw' for each voice a numerically defined guide curve across the sequence of triads and choose the note in each triad which lies closest to this curve. Of course, if each vocal line is formed without regard to the others, such a crude method cannot be guaranteed to assign all three notes of the triad, and will produce many faults such as parallel octaves and parallel fifths, and other contraventions of the 'rules' in §5.1. Nevertheless this is a reasonable starting point and should generate the first draft of a short piece or section of music which can be refined subsequently. In my suite of programs this refinement is done by Program 7, §9.

As an example, an arbitrary run of Program 1 has produced this sequence of 24 triads:

I IV I V I vi IV I vi I IV I V vi ii vii^o V ii vii^o I IV vi V I .

Whatever its merits, let us accept this for the time being and fit four vocal lines. This can be done by taking each chord type – I, ii, IV, etc.[8] – in turn and filling an array with the notes of that triad

[8] In the computer code they are represented by the numbers 1 to 30 in the notation for triads of Tables 9 and 10.

over 6 octaves. For instance, taking C as the reference tonic, triad ii will have the notes D, F, A placed in each octave from the bottom pitch of a 'cello to the top notes of a violin. (The exception is that I have not allowed the bass line to have the 5th of the triad (making a second inversion) unless it is specifically a Ic or other 6 4 chord. I recognise that this is over-restrictive since second inversions of 7th chords were not uncommon in music from the 18th century, but at this stage 7th notes are not being added to the triads.)

The six bars of minims in Figure 15 illustrate the sort of chords produced if four vocal lines are created independently of each other, each tracking its own guide curve. The alto and bass had parabolic guide curves, Eq 1, the alto with $S = 16$, $P = 4$, the bass with $S = 0$, $P = -6$. The soprano and tenor were guided by a sine curve, Eq 2, with $S = 24$, $P = 4$ for soprano, and $S = 6$, $P = -2$ for tenor. These guides span the piece.

Clearly the sequence in Figure 15 is most satisfactory. There are several types of fault because no constraints have yet been imposed between the voices:

- parallel unisons, octaves and fifth between two or more parts. The seven parallel octaves in the treble clef from bar 4 to bar 6 is a glaring example,
- incomplete triads: no 3rd or 5th in chord as at the first beats of all bars except the first,
- crossing voices, though there are no instances here,
- probably too much stepwise movement in the upper parts caused by clinging too closely to the guide curve. There are only 3 leaps in the whole soprano, and these are just of a 3rd.

The above example sets the scene for Program 5. In Program 5 each voice has its own guide curve, but the voices are correlated. Specifically, to avoid most bad parallels Program 5 determines for each voice not only the note which is closest to the guide curve, but the next nearest. The choice is made between the two according to whether a parallel would be produced or not. The chords are produced one at a time starting from the first triad, and each chord is made with four notes, one added at a time. As each note is added, it is checked for parallels with the existing notes of that same incomplete chord and with the previous completed chord, and the next nearest neighbour note used instead if a parallel can so be avoided. This algorithm makes no changes to the first voice to

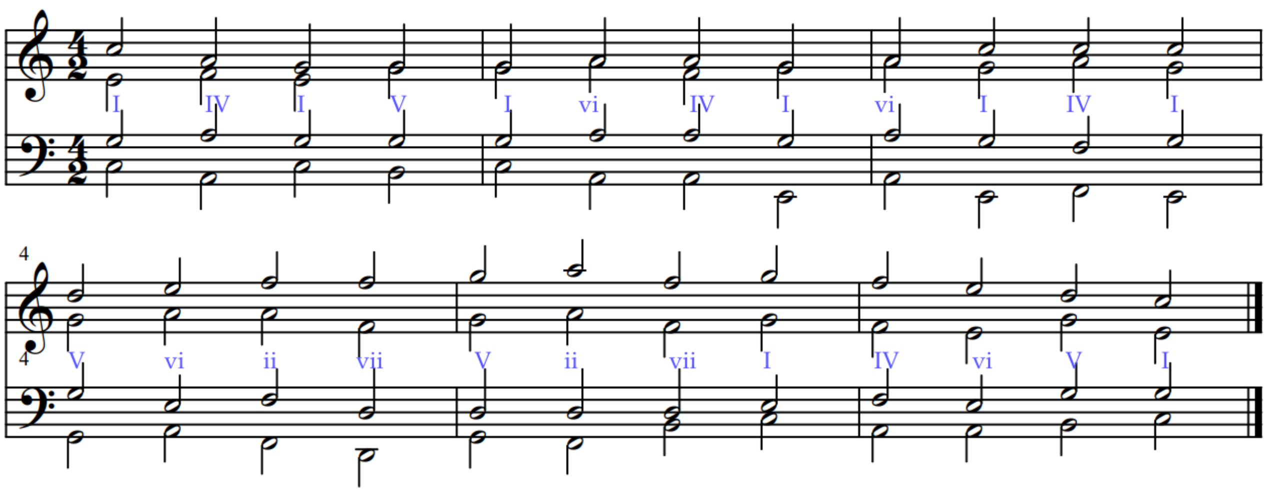

Figure 15: A chord sequence spanned by four un-matched vocal lines. Many errors are present.

be formed nor to any other already chosen. This has the effect of pushing deviations from the guide curves towards the fourth (last) voice to be added to the chord.

Figure 16: As Figure 15, but with first-order correction of parallels applied as the voice lines are being created in the order noted. The guide curves remain the same.

This process makes the pitches in each chord depend on the order in which the voices are added. Compare the A and B versions in Figure 16, which were obtained using the same guide curves and other parameters as Figure 15. In A, in each chord the soprano note was created first, then the alto, tenor and finally the bass. In B the order was the reverse. A has the smoother and better soprano, and B has the smoother though still vigorous bass. Musically A and B are fairly interesting because of the leaps in the lower parts. Because the algorithm does not check backwards, there is always the risk that when the next neighbour is chosen to avoid a parallel with the alto, say, it creates one with the soprano. There is an instance of this in the last bar of the treble cleft of B. Despite this significant limitation, this internal check for bad parallels has clearly made a big improvement on Figure 15. The biggest weakness now is the incomplete triads and this is dealt with by running Program 7 (§9) after Program 5 has produced a 'first draft'. Included in Program 5 are the options to i) choose the order in the which the voices are added to each chord, ii) add the fourth note of a diminished 7th chord to any of the dim1, dim2, dim3 triads; that is, add F♯ to dim1, G to dim2 and A♭ to dim3, and iii) aggregated adjacent beats into longer notes when the beats have the same triad.

6.2 7th and second inversion chords

The algorithms of chord generation described above do not allow for either 7th chords or triads in second inversion, even though these are common in music from the 18th century (see also §7.5 below). Textbooks such as Forte's explain that these chords are produced essentially by melodic processes, specifically passing notes and suspensions. The classic way of generating the dominant 7th chord is shown in Figure 17a where, in C major, the F passing note between G and E lasts for a full metrical unit and thereby attains the status of an independent chord. However V7 must always resolve to I by stepwise fall of the 7th to the 3rd of I. The tritone between F and B resolves by the leading note B rising to the tonic C.

The cadential 6 4 chord is shown in Figure 17b. The addition of the 7th to a second inversion chord removes much of the sense of instability from it. The figured bass notation is then 4 3 (short for 6 4 3). This is described further in §7.5 below, with examples in Figures 52 and 53. At this point we merely note that restricting the software to eliminate all second inversion chords, even those without their 7th, is too severe and does not accord with common practice. Where they are used, however, they must be matched to the adjacent chords by correct voice-leading.

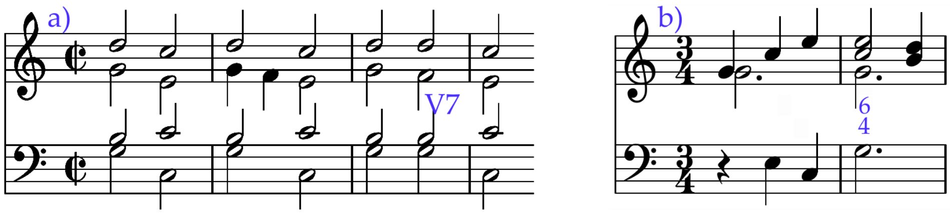

Figure 17: a) Formation of the dominant 7th chord in C major by a passing note. b) Formation of Ic - V progression by suspension from previous chord in an extract from Mozart's piano sonata K332

There is also the passing 6 4 chord, such as Ia - Vc - Ib, or IVa - Ic - IVb. Here the intermediate chord is defined by the stepwise movement of the bass and in most cases the two immediate neighbour chords are root position and first inversion of the same triad. The 6 4 chord is not a stable element in itself, but requires this context.

6.3 Harmonising a given tune

You will have observed that Program 5 creates four voice lines simultaneously. In this way of working any tune to emerge through decoration of the upper line can do so only after the whole piece has been completed in these block chords. In music theory exams two common problems are to harmonise a given short melody, and to add a melody to a given bass, so it is fair to ask whether I have produced programs specifically to carry out these standard tasks. Apart from first species counterpoint (§5.3) the answer is No, at least not until I have explored the possibilities implied by Program 5 and its logical developments. The reason is that neither of these is a trivial challenge and neither has a unique correct answer.

You can take almost any tune as an example and sing it in your head without seeing the musical notation. For a computer to analyse this as a first step to adding a harmony clearly it would need to be given the pitches and relative duration of the notes. From that the algorithm would need to decide which notes belong to the harmony and which are merely surface decoration – passing notes, auxiliary or escaped notes. Even this will depend to some extent on the tempo of the piece, since in a slow piece more notes will generally be given their own chord. Appoggiatura and accented passing notes pose a particular problem since they are important to the tune but not part of the harmony. The algorithm must decide what time signature is implied and where the bar lines fall as the harmony will usually change on accented beats. Also it must decide what key the piece is in. This can be ambiguous. For instance successive notes D and F in a tune may imply the chord of d minor, but also B♭ major, or the dominant 7th on G, or the diminished (B, D, F, G♯). Similar challenges apply in fitting a melody to a given base line. A structured procedure for each type of problem is given in chapter 8 'Composing a soprano voice and harmonising a bass' in the book on Tonal Harmony by Allen Forte.

7 Rhythm and phrases in Western music

In this section we leave computer programs for the time being to examine some structural characteristics of Western music, building upon the principles set out in §2. This will set the agenda for how to take forwards the elementary form- and harmony-generating programs of §8 to §10. Bear in mind that the programs in Group 1 are concerned with the structural aspects of music, while those in Group 2 address surface detail. So far the programs, 1 to 5, have produced only a progression of notes and 2- or 4-voice chords in even minims. Since rhythm, counterpoint and the elaboration of melody are also essential, we examine the options for imposing meaningful rhythm upon the bare tread of chords.

It is not possible to separate melody from its rhythm and phrase structure. The phrase is the shortest element of musical structure which has both a harmonic and a melodic aspect. Phrases are perhaps simplest in songs where one phrase broadly corresponds to what an untrained singer can sing in one breath. There is a lot of symmetry in simple musical pieces, and phrases are often paired, then two or four assembled into musical sentences. One or two such sentences can constitute a section of a work in binary form, closing on a clearly heard cadence. From a structural point of view we are interested in the typical length of phrases and the number of distinct harmonic triads they contain. In §11, which deals with surface detail, interest will move to the number of rhythmic motifs typically found in pieces of different character, and to the shape of phrases – upwards or downwards scales, arching, arpeggios, etc. Phrases and the balance amongst several phrases are important in determining whether a melody forms a memorable tune. I have presented evidence on what makes a good tune in the companion article of that title on *www.mathstudio.co.uk*, and summarise the results in §11.5.

'Rhythm' means how features of the music change with passing time. We recognise at least three aspects:

- the background beat or pulse of the music,
- the melodic rhythm of the surface decoration of the vocal line,
- harmonic rhythm – the varying pace at which the harmony changes.

Many engaging pieces of music have a steady regular 2, 3 or 4 beats in the bass against which a fluid, irregular, almost improvised melodic line moves in the treble. Hymn tunes have a steady harmonic rhythm in which the harmony changes on each beat. Creating a more interesting harmonic rhythm involves identifying the cadences and hence dividing the melody into 2, 3 or 4 bar phrases with repose at the end of each. At its simplest melodic rhythm can be obtained by breaking block chords into broken chords, as in the C major prelude of Bach's WTC Book I and the Prelude to his first cello suite. Rhythmic development will mean extending the duration of selected 'important' notes, shortening others, and adding decorations such as upper or lower auxiliary (neighbouring) notes, passing notes, escaped notes and suspensions. Indeed, in previous centuries singers and instrumental musicians were expected to extemporise on any melody using a scheme called 'divisions' or 'diminutions'. The upper line in Figure 2 of §2 is a division of the lower. The arpeggiation and scales in the study at Figure 5, §4.1 is another simple example of dividing block chords into a melodic line with rhythmic interest. Figure 5 also reminds us that music requires a judicious mixture of novelty with repetition. Too much invention in the rhythm and the piece sounds jumbled and hard to understand; too little and it can be boring. The repetition of rhythmic phrases in a sequence is a standard tool of the composer.

The analysis of melody and harmony is nowadays often carried out using the concepts of structural notes proposed by Heinrich Schenker in about 1920 and called Schenkerian analysis[9]. Briefly, relatively complex musical works are regarded as the elaboration of fundamental structural triads, usually based around the I - V - I axis. Elaboration can be to one, two of more levels, and the object of the analysis is to peel back the surface decorations to reveal the backbone of the music. Conversely, a basic structure can be elaborated, the bony musical skeleton being clothed first in muscles then in skin, and completed with trills, accents and slurs. An example is shown in Figure 18 where the top line is an extract from the violin part of a sonata in F attributed to Handel. The middle line shows the outer skin of decorative motifs removed and the lower line shows the backbone harmonic progression. Elaboration and expansion such as this is commonplace in keyboard variations and other compositions of the 18th century. The example illustrates in reverse the challenge of harmonising a given tune, mentioned in §6.3.

Figure 18: Extract (top line) from the violin part of a sonata attributed to Handel showing elaboration of arpeggiated chords (middle and bottom lines).

In the following sections I explore some of these possibilities within the limitations of the programming I am using – no powerful artificial intelligence with self-learning. The examples in this section have *not* been produced by computer; they are my own human endeavour, quoted to illustrate the types of rhythmic variations which can be overlaid on the bare skeleton of a 4-part chord sequence. As a starting point we need an harmonically 'correct' version of the chord sequence at Figure 15 in §6.1, so I have run ahead of what I hope the 4-part harmony program might eventually achieve, and produced by hand a more satisfactory harmonisation of the soprano in these figures by adjusting the voicing of several chords. The four parts in Figure 19 are therefore the starting material for this present section. Listening to this, it sounds to be in three sections of eight notes each with a natural cadence at its end; a plagal cadence IV - I at the end of bar 2, a ii- vii^o in bar 4 which sounds like a perfect cadence in the dominant key, and a full close perfect cadence at the end. Each is marked with a double bar line.

7.1 Agogic accent and independence of voices

Early ecclesiastical music had complex rhythmic textures. This rhythmic flexibility of polyphonic singing was in large part driven by the accented syllables of the text. The musicologist R. O. Morris says in his book 'Contrapuntal Technique in the 16th Century' (OUP, 1922, p 72) that the music of that period is

> "'rhythmically far in advance of any music that has been written since. Harmony itself, apart from the simplest concords, was merely a by-product of rhythmical experiment; the whole of 16th century texture is essentially an interweaving of independent rhythms .... Counterpoint *is* rhythm and very little else."'

[9] See for example 'Introduction to Schenkerian Analysis ' by Allen Forte and Steven Gilbert, publ. Norton & Co, 1982.

Figure 19: A revoicing of the soprano in Figure 17b to complete the triads and improve the lower parts whilst keeping the same sequence of chords introduced in §6.1.

This view would seem to be at odds with the later species approach to teaching counterpoint in which the emphasis is very much upon melodic lines combining note against note into harmonic intervals, and probably reflects the predominance of vocal lines in early music which gave way to the preponderance of chord progression after about 1700.

Long held notes were often given to important syllables of the sung text, a device called 'agogic accent'. It is an emphasis on notes due to their being lengthened with respect to the notes just before and just after. Even in instrumental music the longer notes sound more significant. In Figure 20 I have lengthened some chords and thereby expanded the music from 6 even bars to 11 irregular ones. This version has a stately feel and sounds as if its could carry noble worthy words. When there is no text, as here, the obvious places to extend note duration are on i) the first and last notes to emphasise the tonic, ii) at cadences (where rests may also extend repose), iii) at the highest note, and iv) possibly also at the lowest. The notes have been extended in units of the metrical beat of a minim because all significant harmonies are best given at least one full beat.

Figure 20: Agogic accents. Some chords of Figure 19 given longer durations, and rests added, to introduce accent and rhythmic drive.

Figure 21: Elaboration of Figure 20 by adding suspensions, passing and neighbouring notes.

Figure 22: Rythmic independence of voices in a manner similar to that common in the 16th century.

From Figure 20 further rhythmic interest can be added be dividing a selection of beats into crotchets with passing or auxiliary (neighbouring) notes, and by suspending some notes into the next chord. Here a sense of style is needed so as not to overdo these embellishments and lose the harmonic clarity which drives the music. Figure 21 is an example I have made by hand. It shows more rhythmic independence of the voices. Taking things further, the four voices may each be shifted in time and given even more independence, as in the music of Palestrina and other 16th century masters. Such independence is emphasised by having each voice share a short memorable theme – effectively a fugue subject – so that when one of the voices sings this theme, attention is drawn to it. In this style bar lines have little significance and do not indicate stressed beats. As one voice after another sings a shared memorable phrase, so focus shifts from voice to voice and the music, if done well, sounds alive. I do not have the skill to imitate the 16th century masters, but Figure 22, derived from Figure 21, gives a rough idea of what I mean.

7.2 Cadences, harmonic rhythm and form

A cadence is a type of musical punctuation, traditionally characterised by a falling of the vocal line and a lull in the musical pace. Cadences help the listener to sense the structure of the piece. Much of the minimalist music of today avoids cadences altogether and so leaves the listener wondering what the music is about and where it is going. While Renaissance vocal polyphonic music was punctuated with cadences, the rhythmic independence of the voices meant that the accents did not as a rule fall at the divisions of the music as written with evenly spaced bar lines. In contrast most dance music from the 18th century has an even number of sections each consisting of an even number of bars. This is so the dancers can move in patterns, each short section ending with a brief pause for a curtsy and a smile. Songs also have a pause at the end of each line or pair of lines so the singers can take a breath. In instrumental art music derived from dances or songs, the structure is also generally in regular phrases of 2 or 4 bars, and sentences have an even number of beats. Examples of this kind are the keyboard and cello suites by J. S. Bach. Cadences are therefore closely linked with the form of the piece. Bach's dance movements are all in binary form A-B separated by a repeat sign, with A and B each having an even number of bars. In the English and French keyboard suites 7 of the 12 allemandes have 12+12 bars, and the other five have 16+16, 8+10, 10+10 and 12+16 bars. In major key and most minor key dances the A section ends on the dominant V chord, and often B starts with V.

Any program to create music in a common practice style must be able to punctuate the sequence of triads. In western music from the 17th century four types of cadence have been recognised, each being known by more than one name:

1. the authentic or perfect cadence or full close: triads V - I or V7 - I. The sense of finality is greatest when the I is in root position, the top note is the tonic, and the key is the tonic of the piece. Departures from these criteria weaken the sense of closure so some writers call the cadence 'imperfect authentic' when either the soprano or bass note is not the tonic.

2. the imperfect cadence or half close has progression I - V, or ii - V, or vi - V. It typically occurs at the end of the first phrase of the piece where a pause is made but the music is opened out to signal that it will continue, probably with an answering phrase.

3. the plagal or 'amen' cadence IV - I. This closes a work but with a weaker sense of key and finality than the perfect cadence. It was used to good effect by Sibelius, though it has a rather antique feel, coming from its origins in modal writing. In hymns it is tagged on at the end after the work proper has come to a full stop with a perfect cadence. Some theorists have doubted whether IV - I is indeed a cadence at all, and generally the subsequence I - IV - I is regarded merely as an extension of the tonic chord.

4. the interrupted or deceptive cadence, V - vi in major or V7 - VI in minor. Here vi substitutes for I. The feeling is that the music has suddenly turned sideways away from the expected forwards direction. It is a common and powerful device for stalling the listener's expectation so that the music can be picked up again and extended for several more bars.

Amongst the simplest forms are the minuets of minor composers – minor because their language is generally simple, even naive. I give two examples in Figures 23 and 24. One common structure is for the A and B sections to be made from 4-bar sentences, so that A will be $a + \hat{a}$ where $\hat{a}$ is almost an exact copy of a except that a ends with a half close and $\hat{a}$ with a full close, often on V. Section B then has structure $b + \bar{a}$ where $\bar{a}$ ends with a full close in the home key. In writing a computer program we have the option to repeat short sub-sequences of triads, changing their cadences, to create such simple pieces.

Figure 23: Minuet from sonata Op 12 by James Hook, 1746-1827. It has an $a|b\ a$ structure.

To produce music in this style a time signature must be specified and the accents arranged with the bar lines. In triple time the first beat is strongest, the 3rd next and the 2nd least. In 4 time the first is strongest, the third next, and the 2nd and 4th equal least. Hence if we have a sequence of triads in Roman notation to parse, the last (second) note of every cadence will occur on the first beat of a bar and be held for at least 2 beats in 3 time, and 3 beats in 4 time. Moreover, if a dance movement is being created, the triads between cadences will have to be adjusted in duration to give a regular 2, 4, 6, or 8 bar structure[10].

Let us see how the example sequence we are using could be punctuated as a minuet in 3 time. Every chord must last at least one beat, unless part of a dotted rhythm when it could be reduced to a half beat. The sequence introduced in §6.1 is

$$\text{I IV I V I vi IV I vi I IV I V vi ii vii}^o\ \text{V ii vii}^o\ \text{I IV vi V I}$$

Here is it again with the potential cadences marked. Notice that I have not treated the two occurrences of I - IV - I as implying a plagal cadence.

$$\text{I IV I } \text{V}_{half}\ \text{I}_{full}\ \text{vi IV } \text{I}_{plagal}\ \text{vi I IV I}$$
$$\text{V}_{half}\ \text{vi}_{interrupted}\ \text{ii vii}^o\ \text{V}_{half}\ \text{ii vii}^o\ \text{I}_{full}\ \text{IV vi } \text{V}_{half}\ \text{I}_{full}$$

Working on 5/4 triads per bar, these 24 chords would occupy 19 or 20 bars. Suppose we aim for an A-B structure of 8+16 bars. The extra 4 bars needed could be obtained by repeating the first 4 triads. This suggests that we split the triads as follows:

[10] The Tempo di Minuetto movements by Haydn show this composer's genius for departing from and extending this simple dance.

Figure 24: Triple time movement from sonatina Op 55 by Friedrich Kuhlau, 1786-1832. It too has an $a|b\ a$ structure.

(I IV I V_{half}) (I IV I V I_{full})	4+4 bars
vi IV I vi I IV I V_{half}	8 bars
vi ii vii^{o} V $ii_{half\ in\ V}$	4 bars
vii^{o} I IV vi V I_{full}	4 bars

In line 3 the sequence vi - ii - vii - V can be used to make a temporary modulation to the dominant key, with return in line 4 via $(\mathrm{I - IV})_V$ becoming $(\mathrm{V - I})_I$. In 3/4 time two adjacent triads will either take dotted minims over 2 bars, or a minim plus a crotchet in one bar. The shape of this short piece would be helped if the melody in the last line resembled that in the first.

7.3 The length of phrases

The examples by Hook and Kuhlau in Figures 23 and 24 illustrate how often harmony changes. If we count 6 4 Ic chords as being essentially V, one triad often lasts for a whole bar, or else for the first two beats of a bar. In a preliminary survey I examined a few pieces in 2 and 3 time and found the number of triads per bar to be between 0·8 and 1·8, with the average at about 1·2. If a phrase lasts 2 bars (see below), this equals between 1·6 and 3·6 triads per phrase.

In a more recent and detailed assessment, I have counted the number of beats in each of 220 phrase samples from 38 pieces of music. Most are quoted in 'What makes a good tune on' at *www.mathstudio.co.uk*, with additional ones by Couperin, Haydn, Mozart, Beethoven, Schubert, Chopin and Ireland. The results are plotted in Figure 25 for pieces in duple time (2/4 or 6/8, green), triple time (3/2, 3/4 or 3/8, blue) and quadruple time (4/4, red). In simple time the beat is one crotchet, and in compound time one dotted crotchet. The graph for duple time (green) peaks strongly at 4 beats (2 bars) and much less frequently at 8. In 4-time (red) there is only one strong peak at 8 beats (2 bars) per phrase. In triple time (blue) 6 beats (2 bars again) is most common, though 12 beats is not far behind. If these peak positions are divided by the respective number of beats in a bar, the normalised curves line up fairly closely onto a collective graph which peaks at 2 bars per phrase. This is confirmed by an alternative calculation in which I found the median (middle value) and mode (most common value) of the number of bars per phrase, and the number of beats per phrase. The median and mode of bars per phrase are both 2·0. The mean and median beats per phrase are both 6·8 and the mode is 8 ·0.

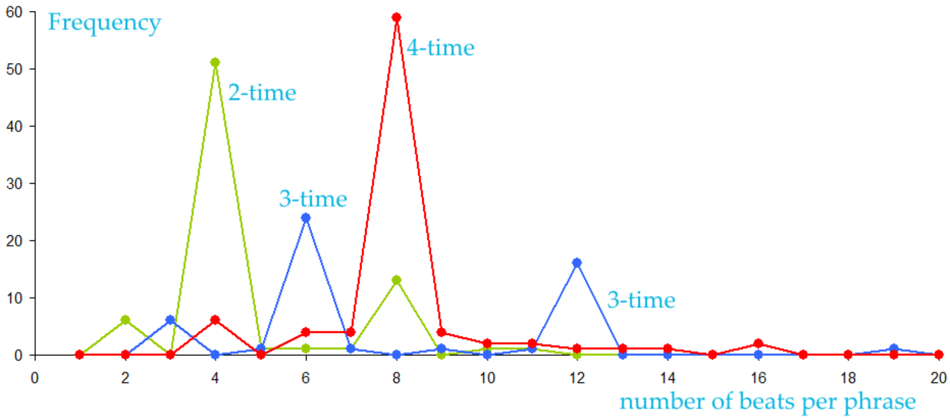

Figure 25: The frequency of occurrence of phrases with different number of beats.

A few subtleties must be pointed out. In most cases there was little ambiguity in recognising a phrase and I am confident in the numbers above, though some phrases have a two-part structure where you can sense a division yet hear it as one phrase. Two fairly common devices for lengthening a phrase are to repeat a couple of bars of it (e.g. Brahms, waltz in A♭ Op 39), and to insert some bars in sequence (song Sweet Nightingale). Rarely there are some very long phrases as in the theme from the last movement of Rachmaninov's second piano concerto.. Where they do occur, they typically follow shorter phrases which have built up musical tension.

Regarding the harmonic structure of phrases, they contain typically 2 to 4 triads. As statistical evidence, I counted the number of triads in 47 phrases from 20 works by classical composers. These occupied 352 beats and contained 158 triads. This is an average of 7·5 beats per phrase, consistent with the separate survey described above, and has on average $2\frac{1}{4}$ beats per triad. The average number of triads per phrase is $3\frac{1}{3}$. This is not inconsistent with the value quoted above of 1·6 to 3·6 triads per phrase for pieces in 2 or 3 time only. Common sequences are I - V (2 triads), I -V - I and I - IV - I (three triads), and I - IV - I - V or I - V - vi - V (4 triads). Very few phrases have just one triad; in ones that do, the phrase is just a broken chord, though 'just' is inadequate when the arpeggio opens Beethoven's third symphony! All this points to memorable musical phrases being harmonically short and simple. I found in examining catchy tunes in major keys that the harmonic basis is almost invariably the primary triads I, V, IV. A common structure for stitching phrases into a musical section is $a\,b\,\hat{a}\,c$ – see the opening of Mozart's A major piano sonata, K331.

7.4 Musical sequences and linear intervallic patterns (LIP)

Sequences and linear intervallic patterns (LIP) are both common musical features which involve repeating a pattern of rhythm and pitch at different scale degrees. They are both important aspects of phrase structure. However, Allen Forte[11] distinguishes them. He says a sequence is a melodic decoration confined to *one* voice. A linear intervallic pattern is a more extended design made up of successive recurrent pairs of intervals moving in parallel formed between the *two* outer voices. The sequence rests upon an underlying harmonically driven chord progression. In contrast, a linear intervallic pattern is driven mainly by horizontal scale-like motion, almost irrespective of the harmony of the chords so created. This is therefore an essentially melodic process. I find Forte's distinction too black-and-white, and will use the term 'sequence' more loosely to describe scale-wise motion with repetition of rhythm and pitch pattern. Moreover, I allow intervallic patterns between any pair of voices, not just soprano and bass.

Sequences are commonplace throughout western music and particularly in baroque styles. We will look at them further in §11, but here consider the examples from Arne, Bach and Vivaldi in Figure 26. Example *a* by Arne is not quite a strict sequence but the ear hears it as such. A similar departure from strict imitation occurs in the full last bar of *c* by Vivaldi, as the sequence closes. There are many examples like this where a pattern is repeated with slight variations to accommodate the harmony. Examples *e* and *f* are from the slow movement of Bach's d minor double violin concerto; both show exquisite pairing of two motifs in sequence. *g* is from the f minor prelude in WTC Book II and shows three voices each with its own motif. At the end of that line I have indicated the repeated unit where you should note the suspension of some voices with respect to the others. This is apparent in the simpler case of two staggered scales at *b* by Arne.

Linear intervallic patterns usually form a chain connecting two principal harmonies: for instance I and V, V and I, Ia and Ib. Take for example *g* in Figure 26; it starts with a broken chord of A♭7 and ends with essentially the same chord in root position. The chain of chords between is a musical insert which has prolonged the A♭ chord over several bars. The function of a linear pattern is to separate yet connect two harmonic pillars of the music. LIPs, therefore, are a sophisticated development of the musical scale used to extend the fundamental harmonic framework. However, they can be overdone. Vivaldi and his contemporaries rarely allowed a sequence to run for more than 3 cycles, though another sequence may start a couple of bars after one has ended. Bach's Prelude in A♭ from WTC II is 77 bars long, and in it I recognise eight sequences, some just two cycles of 1 bar each, some 3 cycles of 1 bar each, some three cycles of 2 bars each, plus much other imitative writing. Following Forte and Gilbert's thorough account in their 'Introduction to Schenkenian Analysis', I list in Figure 27 the skeletons of the main intervallic patterns. Forte labels them by numerals which measure the intervals, so we have 10-10, 5-6, 8-5, etc. These are examined in more detail in the context of computer algorithms in §12. The two marked 7-10-7-10 are beautifully interwoven – see how in one chord the 7th is between voices 1 and 3, the 10th between voices 2 and 3 while in the next chord the 7th is between 2 and 3 and the 10th between 1 and 3. In all cases involving a 7th, the 7th note falls.

[11] Chapter 4 of 'Introduction to Schenkerian Analysis' by Allen Forte and Steven Gilbert, publ. Norton & Co, 1982.

Figure 26: Examples of sequences from the simple to complex. g is in intervallic pattern whose colours are explained in §12.2.

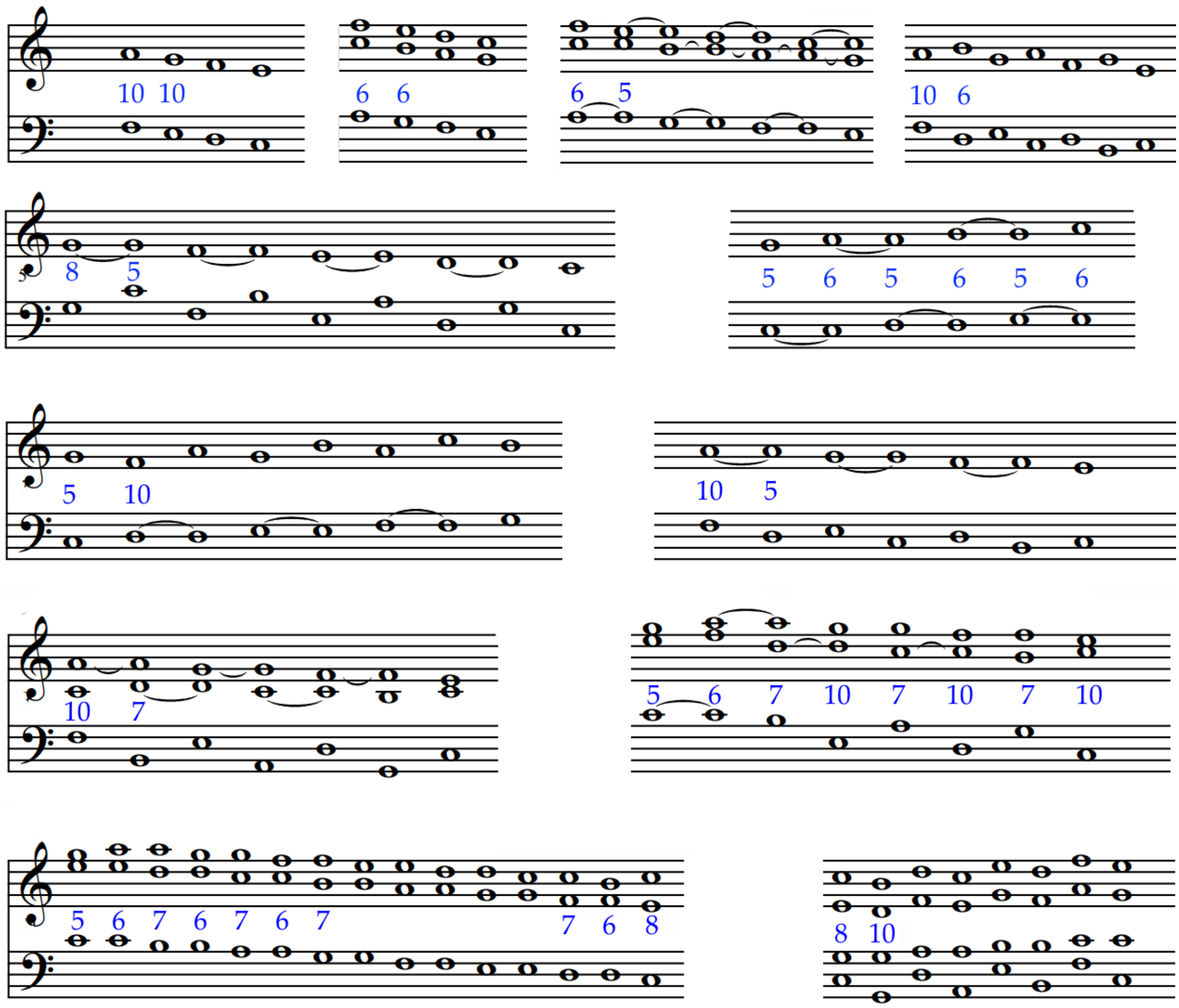

Figure 27: The skeletons of 12 types of linear intervallic patterns as identified by Forte and notated by interval. Patterns 5-6, 10-5 and 8-10 rise, the others fall.

8 Program 6: cadences, modulation and form

Program 6 essentially implements the method illustrated in §7.2 dealing with cadences, harmonic rhythm and the overall framework of the piece. It also is the program which carries out modulation of key, the principles of which were described in §4.3. In practice it is the second program in the suite to use, as it takes as its input data a triad sequence in major or minor generated by Program 1. It therefore steps over Programs 2, 4 and 5 which deal with melodic lines singly (Program2) and in combination (Programs 4 and 5). Program 6 analyses the sequence of Roman numerals and Table-9 notation produced by Program 1, effects its punctuation into a simple musical form, and allows the user to introduce modulation to related keys by modifying the Table-9 labels of the triads as described in §8.1 below. It is necessary for the user to have at least a rough idea of the type of piece to be created – how long it will be, whether in a major or minor key, what time signature to use, and how it should modulate over its length. (Recall that the probability table for the minor key, Table 10, has transition to the relative major built into it.) Longer pieces could be constructed by running Programs 1 and 6 two or more times to produce sections which start and end in different keys, and which are later concatenated.

8.1 Modulation in Program 6

Before going into all the workings of Program 6 I will describe how it handles modulation. Recall the principles outlined in §4.2. Program 6 modulates by transposing selected sequences of triads in Table-9 notation. Bear in mind that no actual notes have been assigned to these triads yet – we are just describing one abstract triad being replaced by another. Recall also that Program 1 operates in the reference keys of C major or c minor. If we want a sufficiently long section of a piece to be in the dominant key of G, say, it would first have to be created in C then transposed. In Table-9 notation, triad 1 would therefore be replaced by 8 (triad C by triad G), 8 by 3 (G by D), 15 by 22 (d minor by a minor), etc. In principle the program will transpose from C major to any other major key, and from c minor to any other minor key. The transposition formula for triad numbers is:

$$\text{Old triad number} = a, \quad \text{new triad number} = b.$$

$$\text{If } a \leq 12, b = (a+m) \mod 12 \quad \text{except if } b = 0, b \to 12.$$

$$\text{If } 13 \leq a \leq 24 \quad b = (a+m) \mod 12 + 12 \quad \text{except if } b = 12, b \to 24.$$

$$\text{If } 28 \leq a \leq 30 \quad b = (a+m-1) \mod 3 + 28.$$

For modulation up I to V, $m = 7$. For modulation down V to I, or I to IV, $m = 5$.

Similarly to transpose from C to E♭, $m = 3$, and to B♭ $m = 10$. Transposition from major to minor is not catered for, but modulation to the relative minor can be prepared in the last few notes of a section by setting $m = 9$ as if to transpose to A major. The actual section in a minor would have to prepared as a separate section written first in c minor, transposed to a minor from its first chord onwards using $m = 9$, and concatenated with the first section.

If the Table-9 pair ... 8 1.... occurs in a sequence from Program 1, they could be fused into a single G major chord, the section up to and including the 8 being reckoned as in C major and the section from 1 onwards as being in G. If the 8 and 1 are separated by a few chords, the intervening ones can be passed over. In principle, therefore, a modulation of sorts could be effected to any new tonic t wherever the Table-9 pair .... t...1.... occurs. However, as Table 11 makes clear, convincing modulations require both a pivot chord and the dominant chord (better still the dominant 7th) of the new key. For this reason Program 6 allows only those modulations listed in Table 11 or similar ones which have a coherent diatonic preparation.

Consider first modulation to the dominant, I → V. Suppose Program 1 has produced a triad sequence with this run :

....	vi	ii	V	vi	V	I	.....
....	a	d	G	a	G	C	.....

There are two equally reasonable ways to effect a modulation from C to G major here.

Method 1: Transpose up (denoted ↑) from the ii onwards. Fuse the two 'a's and two 'D's, passing over the intervening 'e'. This method makes use of the special properties of the circle of 5ths, but wastes two triads:

In C :	vi	↑	ii	V	vi	V	I	..... : now in G
....	a		a	D	e	D	G	..... : now in G
Fused :			a		D		G	.....

Method 2: Sharpen the ii to II. Transpose up only after the G has been stated. This method makes the most of the II-V relationship:

....	vi	II	V	↑	vi	V	I	.....
....	a	D	G		e	D	G	.....

Consider now modulation back from dominant to tonic, which is equivalent to tonic to subdominant: V → I or I → IV. Suppose we have the following run of triads :

....	ii	V	I	IV	ii	V	I	.....
In G :....	a	D	G	C	a	D	G	..... : now in C
In C :....	d	G	C	F	d	G	C	..... : now in F

As in Method 2 we transpose after the new tonic triad has been stated:

In G :	....	a	D	G	C	↓	d	G	C	..... : now in C
In C :	....	d	G	C	F	↓	g	C	F	..... : now in F

The modulation is enforced by flattening of the leading V triad, V → v, counterpart to the forced sharpening of the new dominant chord on transposing up. Furthermore, it may be desirable to smooth the modulation by inserting an extra triad, vi of the new key, so that in G → C the sequence becomes ... a D G C (a) d G C .... The reason is that the progression C → d is not common, but C → a → d is natural and easy on the ear.

Method 2 with forced sharpening and flattening of dominant preparations is used in the program because it is simpler and more versatile than Method 1. It applies also in preparing modulation to the relative minor where in the sequences (ii - iii - vi) or (vii^o - iii - vi) the iii is sharpened to III. The only other modulations which Program 6 caters for are those in the bottom panel of Table 11, §4.3, both in the minor key and both quite rare.

The program therefore scans the whole given triad sequence for the pivotal run (vi - ii - V) which signals modulation up I → V, and also scans for its return via the pair (I - IV). The user is offered the option to place the section between these two into G major. It is also straightforward to insert a modulated passage into one which is already modulated, to give an inner section two or more keys removed from the home key: for instance, A major within D major within G within C. Taking the two pivotal runs in the opposite order of (I - IV) then (vi - ii - V) will take the section between from C to F major. The scan will also flag up (ii - iii - vi), (vii^o - iii - vi) and (vi - iii - vi), any of which could prepare modulation to the relative minor. In a minor key piece modulation to the relative major (or should I say drifting in and out of the relative major) is built into the sequence generator in Program 1. The minor key pivots scanned for in Program 6 are (dim1 - ii - v), (dim1 - II - v), and (v - i - iv). In each case transposition starts immediately after the new tonic triad has been stated.

8.2 Outline of Program 6

While developing the program I took for simplicity that a piece would be built from a number of blocks of 4 bars each, each ending with a cadence. Between cadences will generally be an even number of phrases, but structuring the phrases will largely be left to Part 2 of this article. It would not require much programming to change parameters so that the blocks had another number of bars in each, and that this number could vary from block to block. The program runs semi-interactively. Its steps are as follows:

1. Decide how long the piece (or section of a longer piece) will be in terms of the number B of blocks of 4 bars each.

2. Decide the time signature, T beats per bar.

3. Decide on the density of harmony changes. This will be a value typically between 1 per bar for quickly paced pieces and $1 \cdot 8$ or larger for slower paces. A useful general value might be $1 \cdot 25$. Evidence of the number of triads in a phrase is given in §7.3.

4. Use Program 1 to create a sufficiently long sequence of triads between B and $1 \cdot 8B$ in number. The output of Program 1 will be a string of integers representing triads according to the notation of Table 9. For instance, 1 = I, taken to be C major in the reference key, 8 = V = G major, 13 = i = c minor, 15 = ii = d minor, 28 = dim1, etc. Place this output sequence of Program 1 as a DATA statement or other manner of input to Program 6.

5. Run Program 6. This will first flag up those positions in the sequence where pairs of adjacent chords indicate that some type of cadence could potentially be placed. For each block of 4 bars the program chooses that pair of triads which form the cadence closest to the end of that block, though the user can alter this. The second chord in each cadence pair will fill the fourth bar of the respective block. Following the description in §7.2, the program will flag up plagal cadences IV - I, but will regard I - IV - I as just a prolongation of the I triad.

6. The program then fits the remaining triads (those which are not the closing triad of a cadence) into the intervening sets of 3 bars in each block. The duration of each triad is at least one beat, but can be a whole bar. (I have not yet allowed triads to run for longer than one bar.) The allocation of triads to beats in done be selecting them at random subject to all $3T$ beats being filled by the triads in the given order.

7. Introduce modulation if so desired using the algorithm in §8.1 above. The program locates natural pivot chords and lists them to screen. The user makes a selection, modulating one section of the triad sequence at a time, with the option then to modulate a further section. Two or more nested modulations can be made. By setting the start of the modulation to position 0, the opening key of the piece can be altered. Similarly, by setting the end of the modulation to the last triad position the piece will not transpose back.

8. Should the transposition in key change create positions where two adjacent triads have the same Table-9 number, the program will scan to detect this and remove the duplicate, shunting the subsequent triads forwards by one place.

9. The program keeps track of modulations by entering into a linear array M the prevailing key of each triad on each beat, in Table-9 notation. M is used in Programs 8 to 11 which add melodic decoration and therefore for which the prevailing scale must be known.

10. The output from Program 6 is essentially a harmonic rhythm, written as triads in Table-9 notation, one triad per crotchet beat, plus the array M which records all modulations.

An example will help. This is for a piece of 4 blocks of 4 bars in triple time, lasting 16 bars, 48 beats. Using a harmonic density of $1 \cdot 25$ we need a sequence of at least 20 triads. Here is the sequence of 23 triads created by Program 1. I give both their Table-9 and Roman letter notations:

1, 6, 1, 8, 1 : 22, 15, 8, 1, 6, 1, 8: 1 22, 8, 1, 6, 8, 1: 6, 15, 8, 1
I IV I V I : vi ii V I IV I V: I, vi V I IV V I: IV ii V I.

Program 6 will ignore the IV - I part of I - IV - I (first three triads) and writes to screen the 8 possible cadences – 2 imperfect ones and 6 perfect ones. If the 4 blocks each contained almost the same number of chords, the four cadences would close at positions 6, 12, 18 and 23. Given the actual sequence, the selected cadences pairs are at (4, 5), (11, 12), (18, 19) and (22 and 23). The closing chord of each will occupy one whole bar. I have indicated the ends of each cadence with a colon : in the above list. Next, the screen displays that the modulatory triplet vi - ii - V (22, 15, 8) occurs over positions 6 to 8, and that the return I - IV is possible at (9, 10), (16, 17), and (19, 20). The user is asked to select, so I chose to move to G major from position 6 and return at 16. The sequence after modulation is

1, 6, 1, 8, 1 : 22, 3, 8, ↑ 8, 1, 8, 3: 8, 17, 15, 8, 1, ↓ 8, 1: 6, 15, 8, 1

The transposed section is marked by arrows. Note the sharpening at position 7 and the flattening at 15. The software detects the duplicated triad 8 and removes it, shunting all subsequent triads one position to the left and reducing the total number to 22. The fourth bar of each 4-block is now filled with three beats of the closing triad in the corresponding cadence. The intervening triads are placed in positions 1 to 4 (4 triads), 6 to 10 (5 triads), 12 to 17 (6 triads), and 19 to 21 (3 triads). This is done using a random number to select which bars get more that 1 triad. In the 3 bars of block 1, for instance, the allocation could be (1,1,2), (1,2,1), (2,1,1) triads per bar. With 2 triads in any bar, they can be allocated to beats in triple time as (minim-crotchet) or (crotchet-minim). Even with the same starting sequence, therefore, the harmonic rhythm will vary somewhat each time the program is run. Here is one realisation:

1 1 6 || 1 1 1 || 8 8 8 || 1 1 1 || : 22 3 3 || 8 1 1 || 8 8 8 || 3 3 3 || :

8 17 15 || 8 8 1 || 8 8 8 || 1 1 1 || : 6 6 6 || 15 15 15 || 8 8 8 || 1 1 1||

The numbers represent triads in the Table-8 notation allocated to each beat of 16 bars in triple time. This list is the required harmonic rhythm.

8.3 Realisation using Program 5

The list of triad labels, of course, is not much use in itself; it must be converted into actual chords. Program 5 is therefore used to derive a rough 4-part harmonisation, with first-order checking for parallels between the voices. It can then be refined by Program 7. The output from Programs 5 and

Figure 28: Harmonic rhythm in a 16 bar piece in triple time realised from the sequence of triads, one per beat, output from Program 6. A: voice line guide curves applied after aggregation of triads; B: applied before.

7 is in turn converted to a Lilypond file by Program 3, and from this both a pdf and a MIDI file are produced.

We take the beat to be a crotchet, meaning 3/4 time. Figure 28 A and B are the engraved music corresponding to two versions of the allocation of triads at the end of §8.2. In A, if the same triad appears on adjacent beats in the same bar, the chords are amalgamated into one of longer duration *before* the voice line guide curves are applied. That is, the guide curves are applied only at the beginning of a bar or when a triad changes. In B the guide curves are applied at every beat, which is why the voices are arpeggiated as they try to avoid parallels. In both A and B the guide curves are the same parabola and sine curve as in Figures 16 and 19, and the order of allocating the voices is SABT. Clearly version B has more rhythmic interest and might even be considered the skeleton of a minuet or simple song. The printed score at B also shows the option in Program 5 to aggregate beats in each voice which have the same pitch into longer notes, *after* they have been created as crotchets. If this is not selected, the output has all chords in even crotchets. The smooth, stepwise soprano and the striding bass are both consequences of the way Program 5 places the voice line guide curves across the sequence of chords and pushes the departures from the guide curves onto the last voice to be created, which here is the tenor. Note the F♯s where the piece has modulated to G major. Some chords are incomplete and some parallels remain; improvement to this 4-part harmonisation must await Program 7.

As a further illustration, I created a piece made of 6 blocks of 4 bars in a minor key with no imposed modulation to the dominant. Program 6 also detects cadences in the relative major. The time signature is 4/4. I selected a rather higher average harmonic density of $1 \cdot 5$ triads per bar. There are 24 bars so an input sequence of about 36 triads is needed, though I will not list it. The procedure for creating this piece was: 1) Program 1 to create the sequence of triads as a list in Table-9 notation, 2) Program 6 to punctuate it into beats and bars marked with cadences, 3) Program 5 to realise the triads as 4-voice chords shaped by four voice guide curves, and 4) Program 3 to produce a Lilypond file for engraving the score and creating a MIDI file for playback.

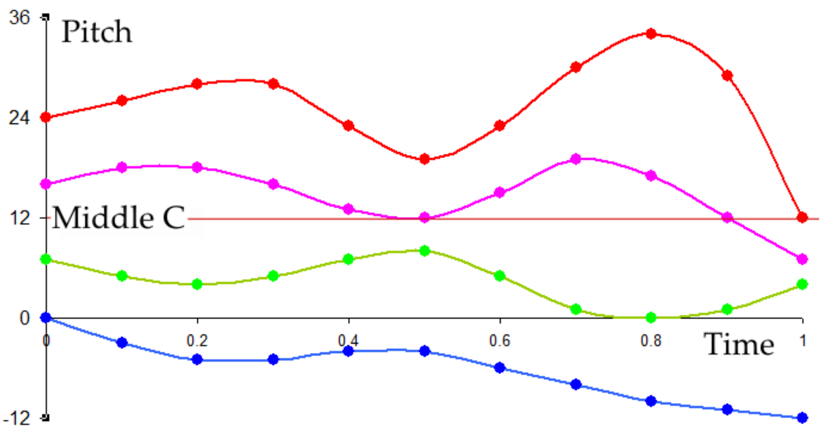

Figure 29: Four voice line guide curves chosen point by point and interpolated with cubic splines. Pitch is in semitones.

For this longer piece I used a different set of voice line guide curves, plotted in Figure 29. The soprano has more maxima and minima as is required for a longer piece. Following the idea mentioned in §5.5, I first sketched the curves on a piece of graph paper to my own design, read off values every 1/10 the way along, and supplied these to the program in a data statement. Alternatively, these global guide curves can be developed in a spreadsheet with graphical display and the values copied to Program 5. To keep things simple the curves here are continuous across the whole music rather than being piecewise at each phrase. More sophisticated guide curves fitted to each phrase are discussed in §11.6. Within Program 6 each curve was scaled in width (time) to the number of triads, and a sub-program interpolated the guide curves' values at each triad in turn using a cubic spline. Clearly the guide curve has less influence on the actual notes for the later voices to be created at each beat, that is, for tenor and bass.

As already mentioned, there is the option within Program 5 to aggregate adjacent beats which are allocated the same triads before applying the guide curves to the voice lines. The difference this has is seen by comparing the top A panel of Figure 30 with the lower B. In A, beats with the same triad (such as 13 13 13 13) have been fused into one long chord, and the guide curves applied only at chord changes and at the beginning of each bar. This realises the harmonic rhythm in the most basic way, and results in a sparse piece with many semibreves. (I only show the first line of this music.) In contrast, for Figure 30 B the guide curves were applied at every crotchet beat, and this has produces more arpeggiation in the lower voices, the order being SABT. Played as it is, Figure 30 B sounds march-like.

The next challenges are to correct the voicing of the chords and to add some rhythmic and melodic interest to the lines, especially the soprano. To some extent this obliges us to think which voices will sing the piece or which instruments will play it.

Figure 30: Harmonic rhythm of a piece of 24 bars in c minor created with Programs 1, 6, 5 and 3, in that order. A: beats with the same triad aggregated into longer notes before spline guide curves applied. B: guide curve applied at every crotchet beat, then adjacent same-pitch notes aggregated.

9 Program 7: 4-part harmony revisited

Program 7 is a large piece of code which, in several stages, aims to produce a 'correct' 4-part harmony from the rough starting version output from Program 5. In this there is no fixed soprano or bass (as in some textbook exercises); rather all notes in each chord may be altered in the search for an acceptable solution. A fully correct solution would require that the voice leading be in accord with common practice, that there be no parallel unisons, octaves or perfect fifths, and all chords are complete, except perhaps deliberately the first and last, and that voices do not cross, except momentarily, and the voice lines do not have unsingable intervals, or run in similar motion at the same interval for more than a few notes. Considering that the harmonisation is only a musical framework which programs in Group 2 will decorate with rhythm and melody, this is perhaps too demanding. I would be prepared to accept some awkward intervals that instruments could play, even if voices could not sing. Though the title of this section is 4-part harmony, it is desirable that the methods should apply to 2 and 3 voices. Indeed, 2-voice harmonisation is closely akin to adding a counterpoint to a *cantus firmus* for which norms have long been established.

9.1 Completed triads and note doubling

The biggest fault in the outputs from Program 5 is that so many of the chords do not present complete triads. Program 7 therefore first checks for incomplete chords and swaps notes so that each chord has at least one root, one third and one fifth. The algorithm is as follows. The program is given as data, in Table-9 notation, the intended triad type of each chord. (This is necessary because if, say, the voices were all on the same note, the correcting program would not otherwise know what changes to make.) The four voices of each chord are examined against triad type, and a record made of the number of roots, 3rds, 5th and, for diminished chords, 7ths. This is compared with a norm and the deficit from the norm evaluated. The norm for major triads is [2 roots, 1 3rd, 1 5th], though [1 root, 1 3rd, 2 5th] is tolerated without change. For pieces is a major key, in minor triads [1 root, 2 3rds, 1 5th] is accepted. The norm for diminished chords is [1 root, 1 3rd, 1 5th, 1 7th]; that is, I have chosen to treat them all as diminished 7th chords for the time being, including the triad vii^o on the leading note. As an example, suppose a major chord has the count of [0, 2, 2, 0] for the number of roots, 3rds, 5ths, 7ths respectively. The deficit would be $[-2, 1, 1, 0]$ and the solution is to move one of the 3rds to a become a root, and one of the 5ths to become a root too. Accordingly, the voices are examined in cyclic order to find the first 3rd or 5th to change. To introduce some variety I used a random number to choose which voice to start with. This means that if the program is run several times with the same input data, the changes to faulty chords may be different. Once a 3rd is found, it is moved down to the root below, 4 semitones for a major triad and 3 for a minor one. The directions of all moves are prescribed, up or down, and generally are through the smallest number of semitones. Checking against the norm continues in a loop until no further changes are needed.

Figure 31 gives an example of the changes made by Program 7 (bottom panel) to a coarse 4-part harmonisation output from Program 5 (top panel). The framework for this music had previously been created using Program 6, which allocated 38 triads (from Program 1) to 8 blocks of 4 bars each. I have marked the changed notes in red. Observe that some bad parallels have been created where there were none before. Program 7 scans for such parallel unisons, octaves and perfect fifths and these are marked in panel B by blue parallel lines. The random choice of starting voice noted above means that some runs with the same data have fewer or different bad parallels from others. Figure 31 B is an advance on A, but even apart from the parallel fifths, it has several ungainly aspects in the voice leading. The order of voice creation in the preceding Program 5 was SBAT, so the alto and tenor will have the most and largest leaps. The guide curves used are those of Figure 29 and have produced a very open texture with almost 4 octaves between soprano and bass in places. Since tenor was allocated last, it is to be expected that most faults involve the tenor. The last three bars are particularly poor because of i) the large E to G leap down in the soprano, ii) the G in the bass of the 6 4 chord Ic is not carried over to the following V chord so the chord resolves in the expected way. It would be better if the bass B and alto G in this penultimate chord were swapped over.

9.2 Correcting bad parallels

In preparation for devising an algorithm for correcting parallels and other voice-leading faults, it is instructive to correct the faults in Figure 31 B by hand. In Figure 32 I have taken the five places where there are seriously faulty chords and given my proposed corrections to the right. Observe that:

1. Some parallel 5ths can be corrected by swapping the notes between the two offending voices, e.g. the bass of bar 25 where tenor C has swapped with bass E.
2. It is better if the bad parallel can be corrected by altering only the two chords involved. At bar 3 is a case where the following chord has been altered too, to prevent another parallel being created and the problem propagating.

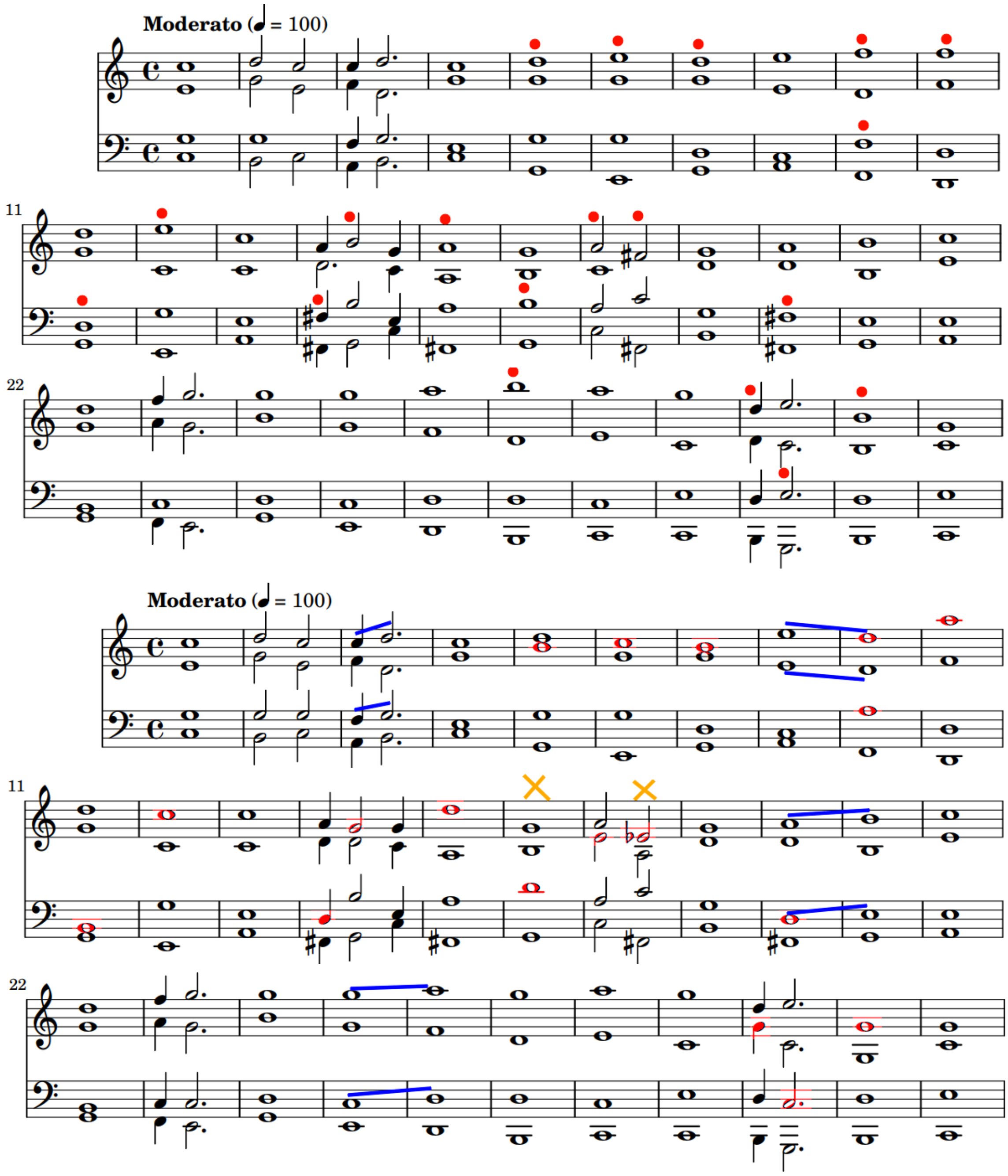

Figure 31: A sequence of chords, in 8 blocks of 4 bars, realised in 4-part crude harmony by Program 5 (A), then processed by Program 7 to complete the triads in all chords (B). Changed notes are marked in red, parallel voices in blue. Orange crosses mark crossed voices.

3. If the chord has the structure [2, 1, 1, 0] for number of roots, 3rds, etc., correcting the fault may involve changing to a [1, 1, 2, 0] structure with two 5ths, or *vice versa*: e.g. i) bar 3, ii) bar 8, iii) bar 16.

4. In problematic cases the structure [3, 1, 0, 0] with three roots could be accepted.

5. Except with diminished 7th chords, the bass should not be swapped with a 5th in another voice as that would create a second inversion; the bass must be a root or 3rd.

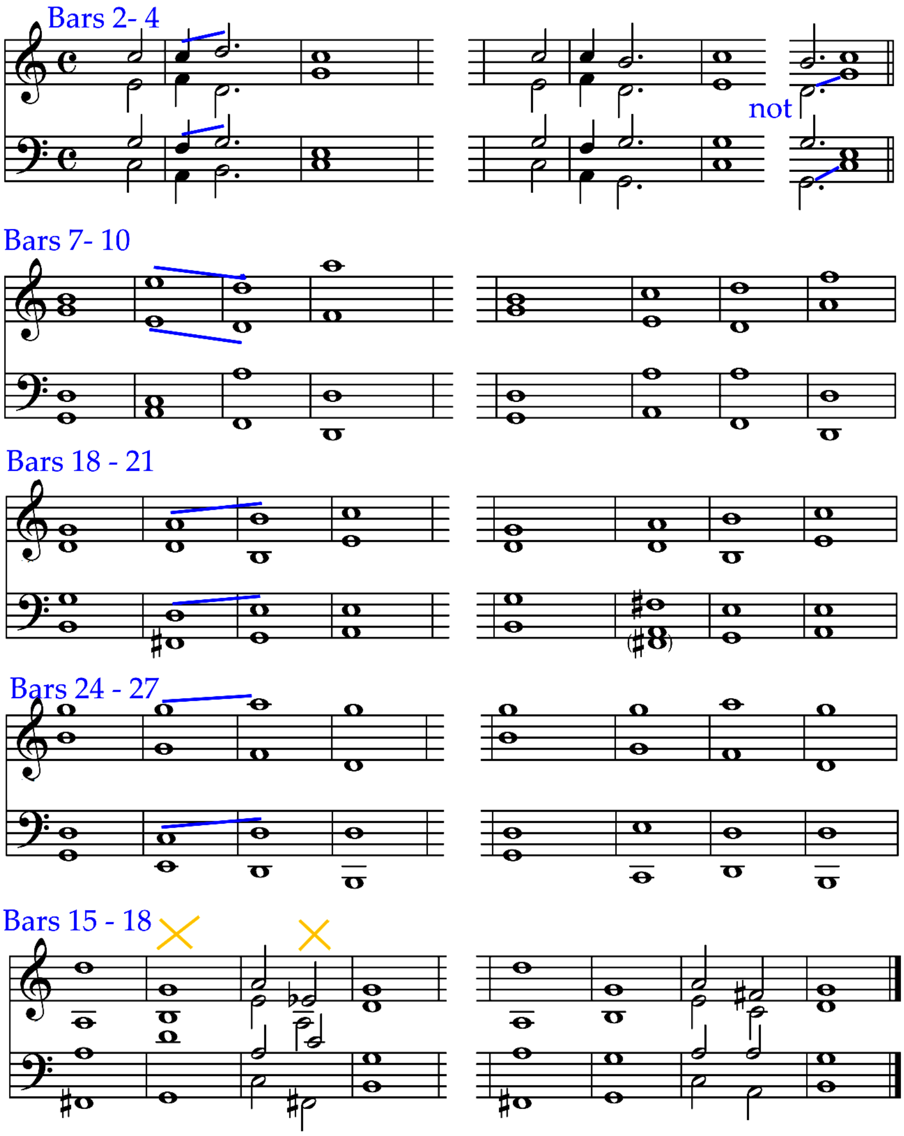

Figure 32: By-hand improvements to the five most faulty places in Figure 31 B.

6. Special chords such as Ic 6-4 chords must retain the dominant in the bass, and other voices must resolved in the standard manner.

7. A passing 6-4 chord is generally acceptable: e.g. bar 19 if A is selected as bass. The alternative of doubling the 3rd (F♯) is also tolerable.

8. The diminished chord need not have all 4 notes, and if only three notes, the leading note (F♯ in the example) should not be doubled.

9. A smoother melody may result from swapping the notes of two voices. e.g. my revised bar 10.

10. Generally leading notes should rise to the tonic rather than leap away. Leaps are better in the bass than in the soprano.

Guided by these points, I developed the next stage of Program 7 to search for corrections of bad parallels by swapping notes. Suppose the two chords which contain the parallels are P and Q, with notes P_0 to P_3, Q_0 to Q_3 over voices 0 to 3. Suppose the parallels are between P_1 and P_2, Q_1 and Q_2. They might be corrected by swapping the notes of P_1 and P_2, or of Q_1 and Q_2. A second stage could take place if P or Q had two fifths, one root; each of the fifths in turn would be changed to a root and the cycle of swapping within each chord repeated. Figure 33 gives the results of this process for the parallel 5ths in bar 3 of Figure 31 B. The program found seven independent solutions, to be compared with the original at the top left. In 1 the F and A in the bass clef have been swapped between tenor and bass, with no other changes. This is the only solution by simply swapping notes. Solutions 2 to 7 have been produced by changing one or other of the D's in the second chord of the parallels to a G – fifth to root. Clearly there is no point in swapping notes if they have the same note letter (e.g. if both are F). Also some swaps are invalid because they would change the chord to a 6 4 second inversion.

It would be crude just to take one of these optional solutions at random and with it replace bar 3. Rather, from this set we wish to choose the best solution, and that brings us to the next stage of Program 7 concerned with the finer points of voice-leading. Solution 1 has the smoothest soprano line, but suffers from the unvocal augmented 4th F to B in the bass. Solutions 3, 4 and 7 have unmusical leaps of a 5th and back in the soprano. 2, 5 and 6 are better, and of these we might pick 5 as best because it does not have the large leaps in the tenor which 2 and 6 suffer from. Accordingly, bar 3 of the piece should be replaced by solution 5. Code has been added to Program 7 to assess the horizontal motion of the four voice lines over the four chords which span the offending parallel plus

Figure 33: Options for correcting the parallel fifths in bar 3 of Figure 31 B, as found by Program 7.

their immediate neighbours. A fault score is allocated to each voice using criteria similar to those listed in §5.1 regarding melody, and the solution selected which has the lowest voice-line fault score. Figure 34 is one example of the corrections thus made to all parallels. A separate piece of code gives partial correction of the cadential 6 4 chord and its resolution in the closing bars. There is some voice crossing at bars 15 and 17, though this could readily be corrected. Clearly, therefore, there is still room for improvement in the voice leading throughout the piece, but the harmony is becoming more correct in a textbook sense.

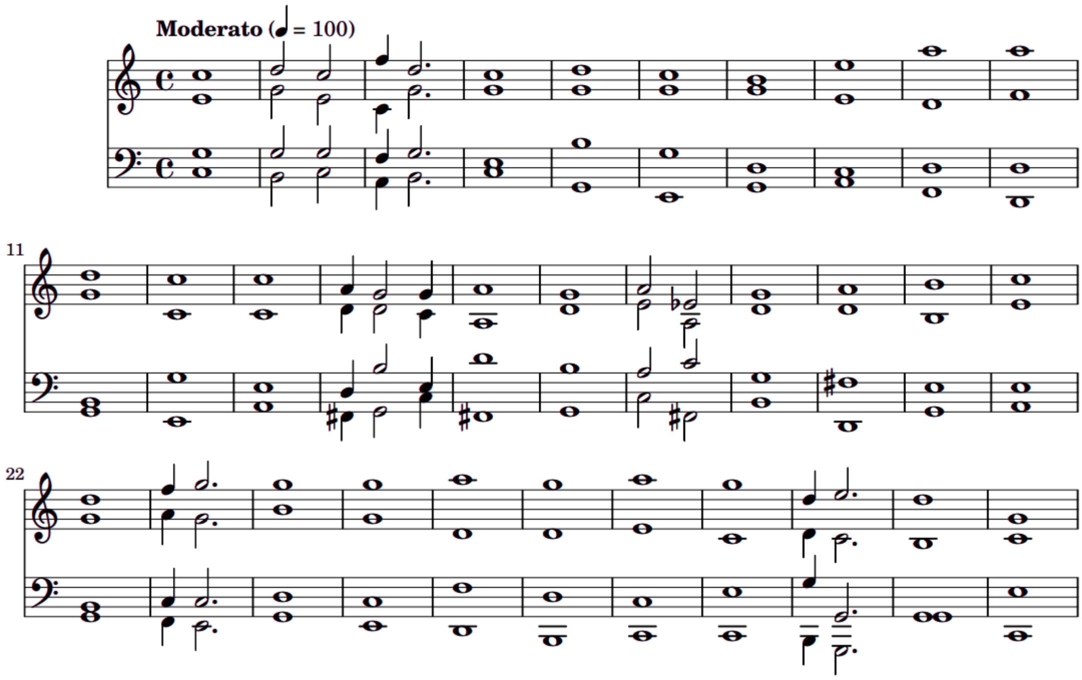

Figure 34: A further stage in the polishing of Figure 31 B produced by Program 7. All chords are complete, there are no bad parallel voices, and the closing cadential 6 4 chord resolves correctly.

9.3 Improved voice-leading

The last stage of Program 7 aims to smooth out the rougher parts of the whole piece. Bear in mind that the overall shape and spacing of the four voice lines has been set by the choice of guide curves such as Figure 29. Also the last voice to be allocated notes in Program 5 is likely to have the most and largest leaps. In Figures 31 and 34 the allocation order was SBAT. A strong bass line is highly desirable and textbooks[12] on 'root progression' advise that strength comes:

- between root position chords where the bass rises a 4th or falls a 5th,
- between root position chords, progression is better if the bass falls a 3rd than rises a 3rd,
- between root position chords, progression is better if the bass rises a 2nd than falls a 2nd,
- if the root falls a 2nd, it is best if the second chord be in first inversion, or both in first inversion.

The program in coded to deal with both cadential and passing 6 4 chords. Having completed all the chords and ironed out all bad parallels, there is not much scope for changing the harmonisation

[12] For example, chapter 14 of Harmony by Thomas Keighley, 1929, pub. Bayley and Ferguson, Glasgow.

of Figure 34 and similar pieces, except perhaps to move some voices up or down an octave, or to make specific adjustments by hand. Accordingly, I have merely written some code which scans the whole piece and allocates to each chord a fault score, similar to that used to select the best option for parallels. The worse parts are thereby identified, with on-screen notice of the type of fault, so that a human musician can assess and decide upon a by-hand adjustment. Hopefully there should be only a few bars which need such intervention.

As an example Figure 35 gives a completed harmonisation which started as Figure 31 A. I ran Program 7 again and edited 4 chords by hand following on-screen prompts from the final stage of Program 7. These edits were made using a simple note editor within Program 7 itself. To help with editing, the program saves the version in its Figure 34 state so that this can be converted to a Lilypond file and hence pdf file so that sheet music can be viewed on screen. Thus while editing the user can refer to the printed music in front of them. Alternatively, and perhaps more conveniently, the music can be edited on screen with Lilypond and then the Lilypond file converted into the program's internal representation based on Table 9 for use in other programs in the suite.

The other changes in Figure 35 from Figure 34 were produced by the software due to the random number method of selecting the voice at which to start swapping notes. I judge that harmonisations such as Figure 35, in 3 or 4 voices, are an adequate harmonic framework, and that further refinement of the algorithms is unnecessary.

We at last have a sufficiently robust framework on which to place melodic decorations using Programs 8 to 11, as described in the following sections.

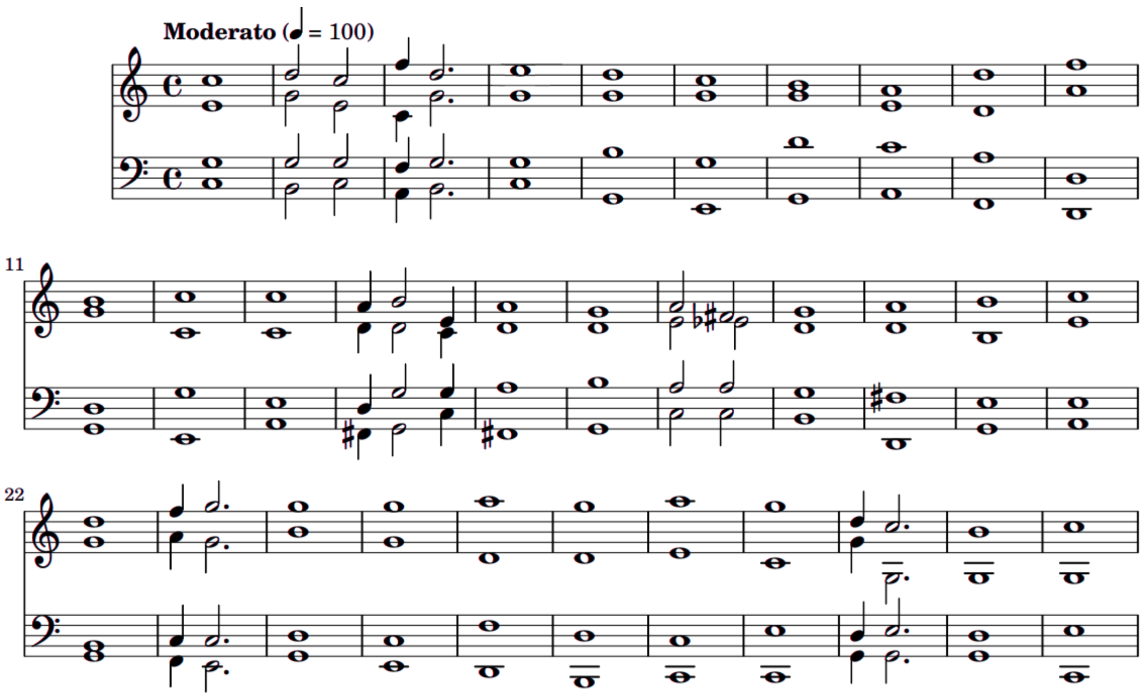

Figure 35: Another realisation of Figure 31 B, with a few by-hand edits to complete the harmonic rhythm in 4-part harmonisation. Some pitches could be moved by one octave to suit the instrumentation.

10 Program 8: linear intervallic patterns (LIP)

10.1 Forte's canonic forms revisited

Sequences and linear intervallic patterns were introduced in §7.3, where Figure 27 listed the canonic forms as identified by Allen Forte. Following Forte we distinguished two main types of scale-like repetitive pattern:

1. One occurs in one voice only and is driven by harmonic progression of triads, as round the circle of fifths. Forte restricts the word 'sequence' to these.

2. The other is driven primarily by the parallel scale-like motion of the melody in two or more voices, irrespective of whether the chords so formed function harmonically. Forte calls this latter type 'linear intervallic patterns' and describes the canonic forms shown in Figure 27.

Whilst a sequence in one voice may be little more than decoration on top of a harmonic progression, intervallic patterns are a powerful tool for expanding the music between critical chords, and so making a place for the development of motifs. I regard the two above devices as extremes of a spectrum. Some scale-wise passages in any number of voices will have an in-between structure and so be driven towards their goals by both harmonic and melodic forces.

The examples in this section will be based mainly on the simple 16 bar piece of 4-part harmony in Figure 36. It was created in the usual way using in turn Programs 1, 6, 5, 7 and finished with by-hand editing of a few notes to improve the shape of lines and the spacing of the voices – mostly transposing the tenor voice up one octave to prevent a muddy sound. The guide curves in Program 5 have been redrawn from Figure 29 to produce a closer voicing, and the voices were added to each chord in the order SABT. In itself these four blocks of four bars each sound like one of those psalm charts traditionally used in Church of England cathedrals, and is typical of the base material on which all of Programs 8 to 11 operate.

Figure 36: A simple hymn-tune-like piece on which to demonstrate some operations of Program 8.

To give some idea of the musical effect of inserting a linear scale-like section into a given harmonic progression, I have *by hand* modified Figure 36 by inserting such a pattern between the red bass notes towards the end. My extended piece is in Figure 37. Note these points:

- I have used the leap of a 4th (here in the bass) as the opportunity to insert two passing notes, each lasting one bar. As luck has it here, there are two adjacent leaps of a 4th, so two intervallic patterns have been inserted.

- Unlike the sequences which occur in a single voice line, the intervallic pattern (LIP) is made from new notes, and so it extends the piece of music by however many cycles are inserted.

Figure 37: As Figure 36, but with two linear intervallic patterns inserted between pairs of chords in bars 12 to 14.

- To each inserted bass note I have added a parallel scale a 6th above, picking up the G in bar 12 and the C in bar 13.
- Two simple rhythmic decorations are used to replace the inserted dotted minims. Because there are two adjacent sequences, I have swapped these between alto and bass in the second sequence.
- The second sequence would strictly end on an F in the top voice, incompatible with a minor, so I have let it ascend to high G. This is fortuitous because it adds the highest note to the piece, making a climax, and also forms an a minor 7th chord which resolves strongly to d minor. It would be better if the d minor chord, bar 15, were now revoiced to place F in the soprano, and the chord in root position.
- Only two voices are involved in the intervallic patterns. The change in texture is welcome relief to the ear and adds to the sense of climax when the full 4-part texture returns.

The net musical effect is a sense of expectation caused by delaying the closing cadence. The placing of the intervallic pattern towards the end of the piece is significant to its effect. We cannot expect such good luck in the placing of chords with every piece, but Figure 37 illustrates one type of musical variation which the suite of programs are coded to create.

Let us reconsider the principle LIPs described by Forte, Figure 27 and developed in Figure 38 below. The most straightforward linear intervallic patterns are based upon the parallel movement of two voices at a separation of a 3rd (= 10th) or a 6th. These are concordant intervals which can move in parallel up or down a scale for several steps without becoming unpleasant. The scale seems to climb or descend towards a goal – the second pillar chord. It occurs in Figure 38a at the interval of 6th. The obvious elaboration is to insert the 3rd to create a chain of descending triads in first inversion, Figure 38b. Forte would point out that these are not functioning as triads in a harmonically meaningful progression, but simply as a decorated scale.

Once we have admitted intervening notes, the gate is open to constructing parallel scales on almost any starting interval. In this way the patterns of c to f are readily made. Some correspond with Forte's list in Figure 27, but my 9-10 and 8-5 are different, though I see nothing wrong with them. The only interval not represented is the 4th, perhaps because it is a rough sounding interval best avoided in exposed places. Almost all patterns share the property that in each cycle unit there is at least one perfect or consonant interval at which the ear has fleeting repose. Some of the sequences – a, b, c, f and possibly also e – are reversible to form ascending scales. d is not invertible because

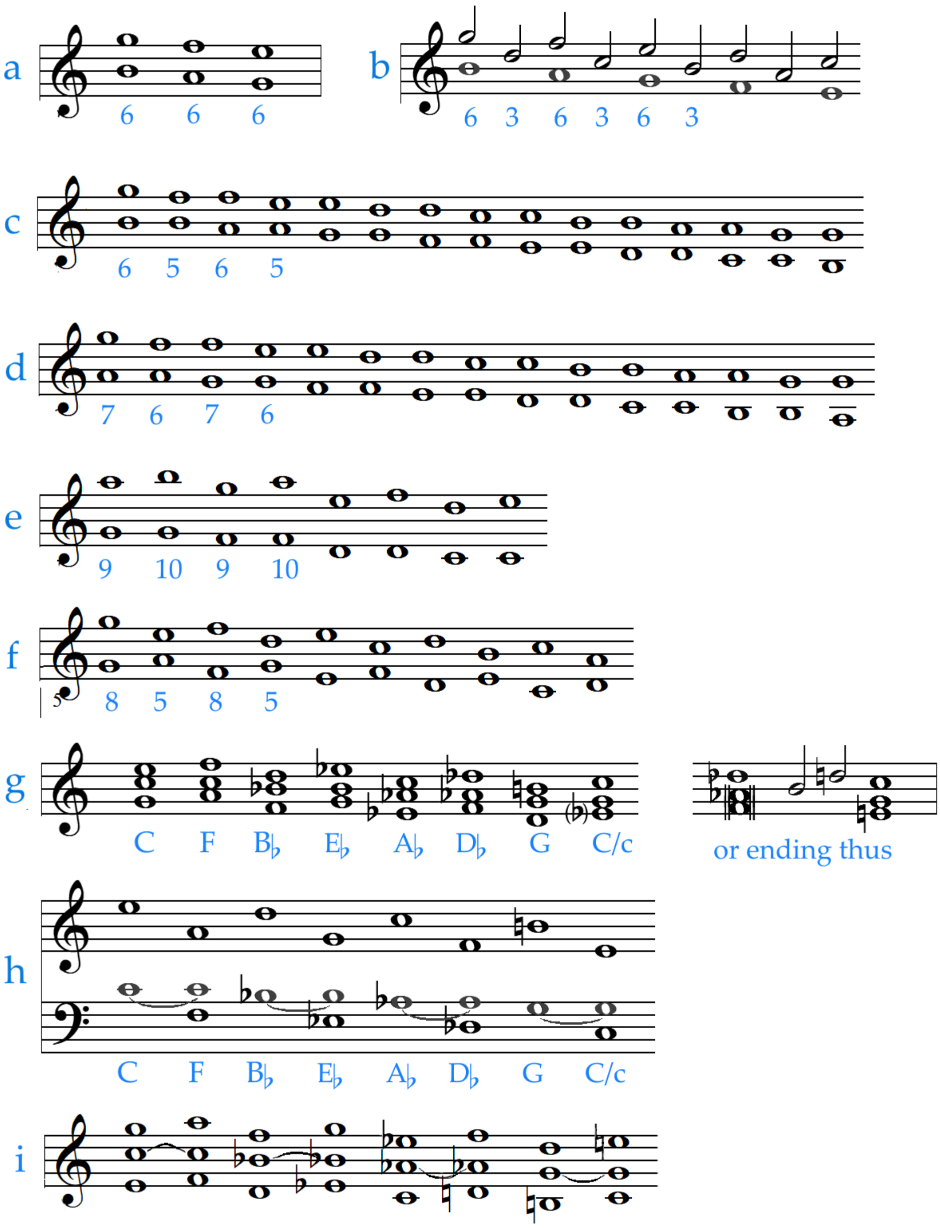

Figure 38: The skeletons of some linear intervallic patterns.

the 7th must always fall. At Figure 38g, h and i are three versions of the same progression which moves round the circle of fifths as C→B♭ →E♭... → *C* as the letters underneath show. This has every look of a linear intervallic pattern, yet in moving V→I at each step (with a glitch at D♭ →G) it is actually driven harmonically as much as melodically[13].

10.2 Examples from the Masters

Because scale-wise repetition has been such a feature of music since the 18th century, and because composers use it with skill and subtlety, I will analyse examples from Vivaldi, J. S. Bach and Haydn. I will use the word ‘sequence’ loosely to include all types of scale-wise repetitive passages.

[13] g, h and i are not reversible, though corresponding reverse sequences can be made with the flattened notes changed to naturals.

Vivaldi : Figures 39 and 40 are two extracts from the slow first movement of the cello sonata in B♭, RV 47 in the Bärenreiter edition. The horizontal green lines mark the pattern units in the two sequences, and the chords are given in figured bass notation. In the first extract the previous chord is the tonic, B♭, so the first sequence bridges I→V with 4 cycles of 9-4, 10-6. The cello part is varied in the last cycle. The chords make an imperfect journey round the circle of fifths, with the obvious glitch of E♭ →a7 instead of A♭, but then the circle continues. It is essentially Figure 38g. So is this driven mainly melodically or harmonically? My answer is 'in equal measure'. The shorter sequence made from a rest and three quavers is a simple melodic decoration of alternating I and V; it appears also in Figure 39. In this second extract the repeated pattern lasts a whole bar and expands I→V in two cycles, B♭7→E♭, c minor7→F, so I→V has been replaced by I→IV→ii7→V, a harmonically meaningful progression. The cello meanwhile merely arpeggiates chords of B♭7 and c minor.

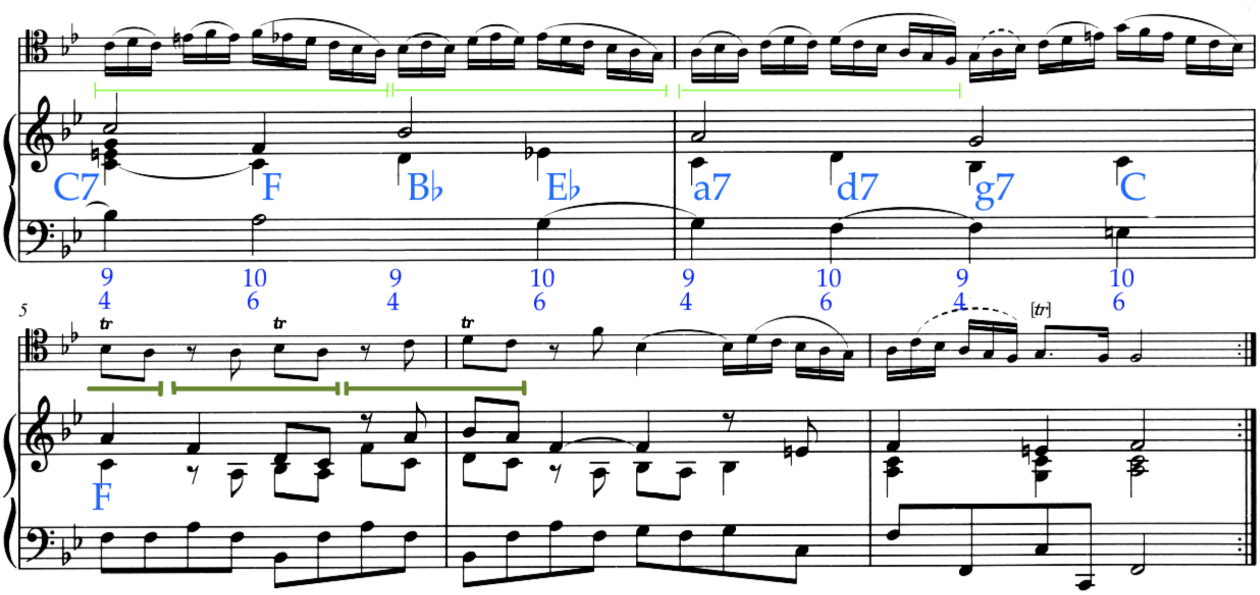

Figure 39: From Vivaldi RV 47, movement 1, Largo.

Figure 40: Vivaldi: from later in the same movement.

Bach : I have already given examples of Bach's sequences in Figure 26 e to g. Example g is a quote from the prelude in f from WTC II. The previous chord (not shown) is E♭ and from there the chords in the right hand are A♭7, f7, g7, E♭7, f7, D♭7, E♭7, c7, A♭7, one per crotchet. The implied chords on each beat in the left hand are different: A♭, D♭, g, c, f, ♭, E♭, A♭, in the progression 6→5→6→ 5... The mismatch of treble and bass is created by suspension of the first two semiquavers in the right hand from the previous bar, as indicated by the blue and green colours in Figure 26g. Bach uses the discords to drive the music forwards, one chord tumbling on top of another. The sequence has 4 cycles of 2 crotchets each and expands A♭, the relative major of the piece.

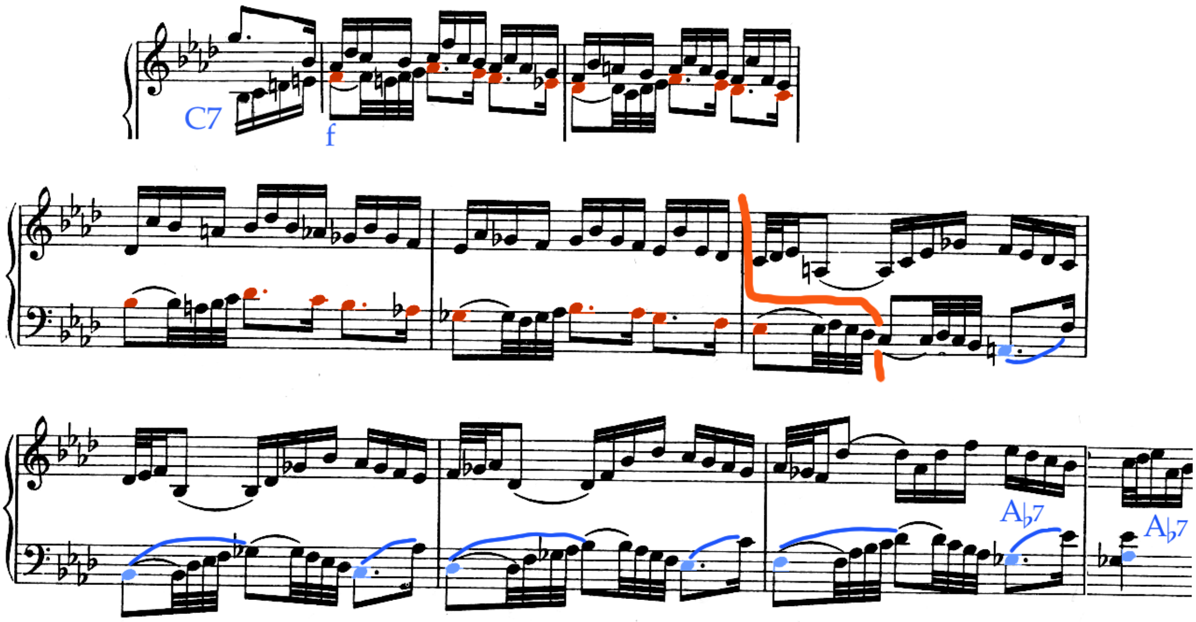

Figure 41: Bars 40 to 48 of Bach's Prelude in A♭, WTC Book II. From the ABRSM edition.

As a further example, Figure 41 presents two of the many sequences in the A♭ prelude from WTC II. In each the pattern length is one bar. In the first the bass descends in steps of 3rd, while in the second it rises in 3rds. Closer inspection shows that the first is built from a descending scale of b♭ in the bass (red notes), and the second from a rising scale of b♭/D♭ (blue notes). Parallel to this rising scale is a second scale a 6th above (blue tie lines). The two sequences join at a diminished 7th chord, marked in the figure by the curved red line. The combined passage links f minor to its relative major A♭, which immediately resolves to D♭ at a cadence in bar 50.

How strict are these sequences? The first observes the same rhythm in bass and treble for 4 bars, but only the bass follows the pitch pattern strictly; the treble has similar 4-semiquaver figures – rise, fall, fall by 1 scale degree – but the intervals are varied to suit Bach's intended local harmony. The second sequence in strict in both hands for 3 cycles, but in the 4th the right hand departs in pitch. Both sequences are driven mainly melodically by the scales in the bass.

Haydn : We examine two pieces by Haydn. The first is the opening of the fugue which concludes the string quartet in A, Op 20 No. 6, shown in Figure 42. This joyful piece is constructed extensively from a linear progression first stated between the two violins at the interval of 3rd or 10th, marked by blue tie lines. In figured bass notation the sequence is simply 3-3-3-3-3-... or 10-10-10-... . The octave transfers add much to the lively character. When the viola enters, the sequence is repeated with the 2nd violin playing the fugue's subject while the 1st violin adds a decorated note (purple phrase) which jumps down in parallel. This phrase is repeated by the 2nd violin when the viola takes up the fugal subject (green tie lines), and the pattern goes on after the cello has entered. I have noted the implied triads from which it is clear that this is not a harmonically driven sequence because the progression does not make much harmonic sense. Rather it is driven melodically, the whole first line moving A→E, I→V. The second line moves back V→I so that the viola's entry is precisely one octave lower than the 1st violin's, and then the cello enters an octave lower than did the 2nd violin.

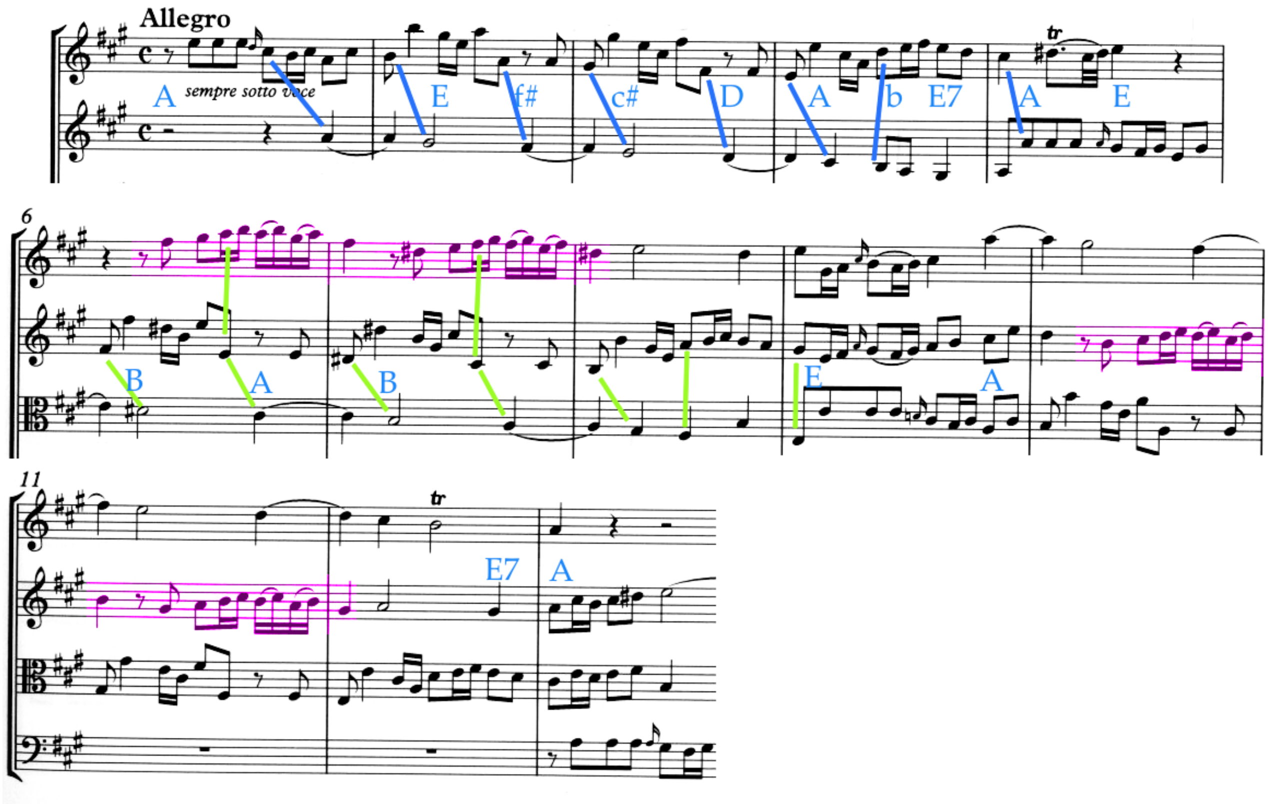

Figure 42: Opening of Haydn string quartet Op 20 No. 6, last movement, a fugue. Edition Peters.

For the second example from Haydn I have chosen most of the development section from the last movement of a piano sonata in A♭ composed in 1786, Figure 43 (from Edition Peters). This remarkable section is built around sequences. The overall structure of the movement is A B $\hat{A}$ where A is a repeated exposition, B the development and $\hat{A}$ the recapitulation. Figure 43 starts at bar 8 of the B section, which starts and ends in E♭, the dominant key of the piece. I have identified 6 subsections, labelled 1 to 6 by the green dividing lines. Of these 1, 3 and 5 are sequences. 2 is a short loop from f minor and back, 4 is a decorated and extended f minor chord, and 6 is an extended cadence closing on E♭ over an E♭ pedal.

Looking at these linear intervallic patterns,

- subsection 1 moves in 4 cycles of 2 bars each from E♭ to f minor over a rising scale in the bass which starts diatonically but ends in chromatic semitone steps. In the last 3 cycles the right hand moves in strict step-wise progression. The harmonic progression makes sense, so we may conclude that this is driven both melodically and harmonically.
- subsection 3 is a loop from f to f in 3 2-bar cycles alternating 6-7-6-7... . Over the first 3 cycles the left hand steps upwards while the right hand steps down. If the first chord of subsection 3 were f7, with f and E♭ in the right hand, the sequence would sound f7→D♭, E♭7→A♭, D♭7→g$^\circ$, g$^\circ$ →C7→f. The diminished g$^\circ$ chord is a harmonic fudge to escape a strict V-I progression, and as a by-product its discord adds bite to the music. Despite the meaningful harmonic direction of V-I in each cycle unit, I consider this to be a linear intervallic pattern which extends the chord of f, the relative minor of the piece: vi→vi.
- subsection 5 starts with 4 bars looping around f which anticipate the sequence. The sequence

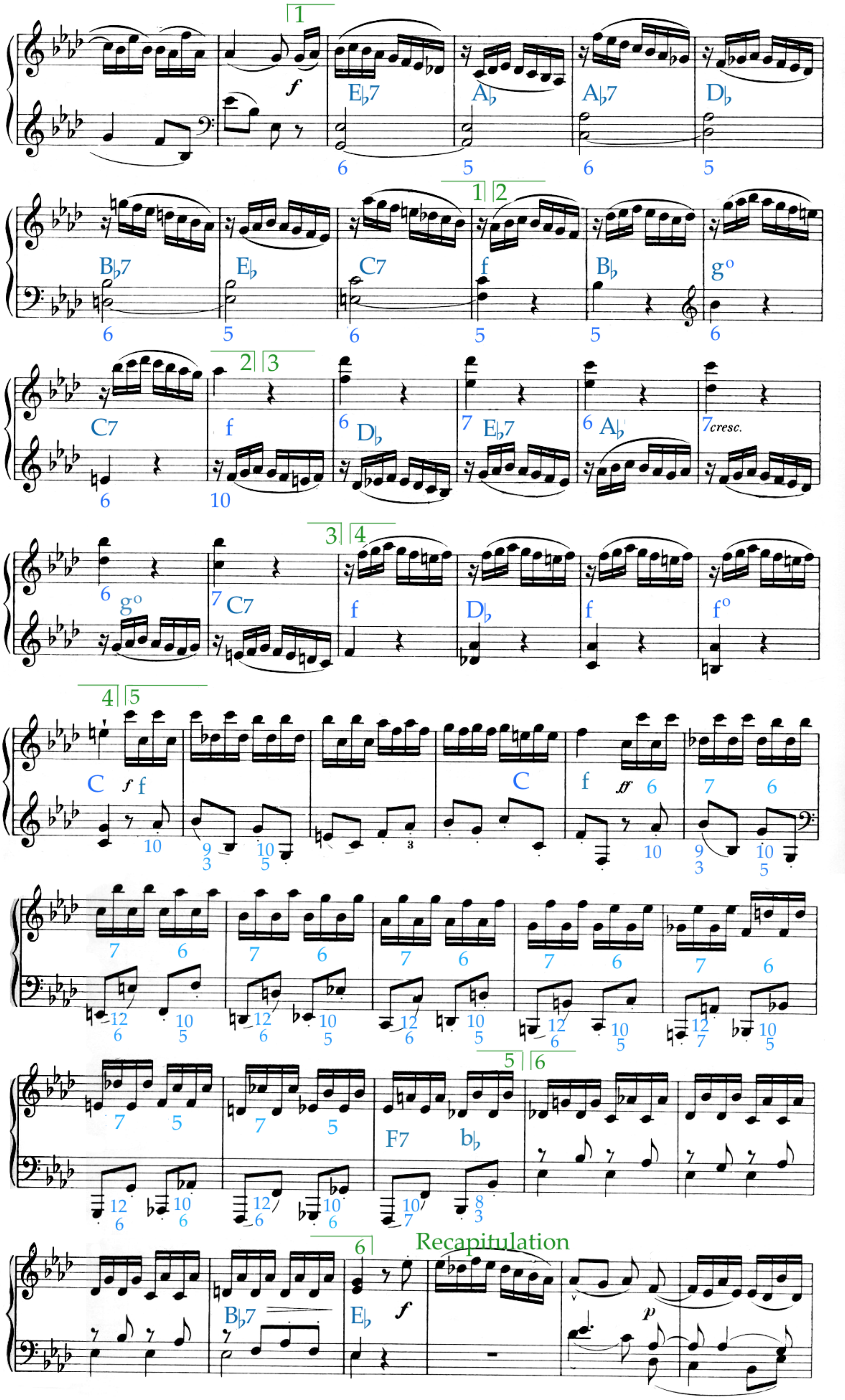

Figure 43: Development section of a Haydn piano sonata in A♭. From Edition Peters Vol 1.

starts properly at the end of the line. In the right hand the pattern is 6-7-6-7-6-7... . while with the bass included the overall chords progress 12-6→10-5 as shown. This is a linear intervallic progression which moves strictly through five 1-bar cycles before harmonic considerations intervene and the pitches are modified within the same rhythm to bring us to the drawn out pre-cadence passage of subsection 6. Subsection 5 moves from f to b♭, a V-I progression though in the minor.

To conclude this analysis from the masters, note also the striking use of sequence made by Rimski-Korsakov in the first movement of Scheherazade, quoted on page 469 of the book by Kostka and Payne. Against rising then falling arpeggios in the bass, suggestive of ocean swell, the chords move C♯ → E♭ → F → G → A → B7, all within the tonality of E. The bass notes are E♯, E, F, G, G♭, G, A, A♭, A, B, B♭, B, C♯, C, B, a remarkable progression by diatonic and chromatic single steps. The musical effect is a sense of expansively moving through an exotic seascape.

From these examples we may draw these conclusions:

- There is a spectrum of influence in sequences: at one extreme is a sequence driven harmonically by the circle of fifths where the repetitive pattern may be confined to one voice, and at the other the intervallic pattern driven melodically by parallel scales in two or more voices. Most sequences in practice owe something to both influences, and one which starts as a pure linear intervallic pattern will often be modified harmonically, though not rhythmically, to bring it to a 'smooth landing'.
- Any scale, particularly in the bass, has a strong directional feel, and this can be intensified when chromatic notes are added to give semitone steps.
- Linear intervallic patterns mostly join two important chords. V→I, I→V, V→V, vi→vi are typical.
- Apart from where the intervallic pattern is actually the opening theme (as in the Haydn quartet fugue), it is unusual for a linear pattern to occur close to the start or end of a piece, or major section of a longer piece. Normally they will be separated by 4 bars or more from the start or end to allow the key of the piece to be established.
- An intervallic pattern usually fits between cadences. It may start just after a preceding cadence, and in general will not continue into the two chords which define the cadence, but end a couple of bars before the cadence.
- Sequences usually maintain the same rhythm even when the pitches are no longer strict, and this rhythm often continues into the following passage, such as one which extends a chord over several bars.
- The cycle unit may be simple, such as in the progressions 6-6-6-6 and 10-10-10, but often the unit is divided into two by a suspension of one voice to give 6-7-6-7-6-7, etc. 7th chords are quite common.
- Discords caused by suspensions or local pitch changes to, say, diminished 7th chords can drive the music forwards.
- It is not unusual for one linear pattern to follow another, perhaps with a few bars decorating one important chord between them.

- Composers will use short passages between or after a linear pattern in which the harmony is essentially static, perhaps by extending an important chord over several bars. Vivaldi often repeats a two or four bar phrase exactly at the start or end of a section. This creates a structure in which important chords are elaborated into stable pillars in the music, joined by a moving chain of chords in sequence.

Linear intervallic patterns are thus a major tool in the extension and development of a piece. They provide musical 'space' in which previously heard motifs can be decorated, varied and extended.

10.3 The algorithm for intervallic patterns (LIP) in Program 8

The function of Program 8 is to insert bars of LIP between selected pairs of chords in the given 3- or 4-part harmonisation. In light of the above analysis, there are several ways in which this can to done, depending on the opening and closing triads, what type of pattern is inserted, the number of cycles required, and how the inserted notes are fitted between the given chords, the latter being governed by the duration of the notes and the time signature. A challenge in writing the code has been to cater for all the different options.

Here is an outline of the stages of the program:

1. The starting place is a complete hymn-like piece in 3- or 4-part harmony, such as Figure 36. The time signature is given, and the 3 or 4 voice lines placed in DATA statements, with each note recorded in the form [pitch, duration], pitch being on the MIDI scale with Middle C = 60, and duration with crotchet = 24 time units, minim = 48. Also transferred from Program 7 are, for each chord: i) the corresponding triad types in Table 9 notation, ii) the positions of cadences, and iii) the local keys after any modulation. This encoding and the local key run note by note in parallel with the four voices, effectively as virtual 5th and 6th voices.

2. The internal notation is as follows. Let $T, 1 \le T \le 30$, be the triad type in Table-9 notation, and let $C = 1$ for the first chord in a cadence, $C = 2$ for the second chord. These are encoded together in one byte as $8T + C$. T is later recovered as $(8T + C)$ DIV 8 and C recovered as $(8T + C)$ MOD 8.

3. These data are read into memory and the MIDI pitches converted into a diatonic scale in the key of C, as also used in creating passing notes in Program 9. In this diatonic scale +2 means 'two scales steps higher' from the note in question in the prevailing key.

4. When a LIP is created, the status of the notes involved is noted by adding to the $8T + C$ code. After the program has created one LIP, it offers the user the option to insert another in a different place. The flags in virtual voice 5 are used to avoid cadences and existing LIPs.

5. The music is scanned again to identify positions away from cadences where an intervallic pattern could reasonably be inserted. For pieces in a major key these positions are

   - at I only, where Ia could be joined to Ib or *vice verse*,
   - at V only, where Va could join Vb,
   - at vi only, where via could join vib,
   - between I-V,
   - between V-I,
   - between ii-V or II-V,

- between IV-I or IV-V,
- between vi-ii.

In a nominally minor key piece cadences can be between I and V in the relative major. The first three types above involve only one chord (call it A), the others two (A and B). The scan excludes the first 2 and last 2 bars of the piece, since here the key should be clear. These optional positions are listed to screen together with information on the A, B triads involved, and the user chooses one. (I decided to give the user the choice rather than have all selections made by a random number generator.) Let the notes on either side of the intended insert be A at position k and B at $k + 1$. If only one chord is involved, it is at k.

6. At the selected position k the intervals between all pairs of voices in A and B are listed – e.g. between (Voice 0, Voice 1), (Voice 0, Voice 2), (Voice 0, Voice 3), etc. Where the LIP is to be inserted between two distinct given chords, the intervals in the second, B chord at $k + 1$ are used to select the type of canonic LIP. Those intervals in B upon which a canonic LIP could close are listed to screen. A LIP must end in one of these intervals: 3rd = 10th, 5th, 6th or 8th = one or two octaves, and could be between any two voices. Typically there will be three or four candidate intervals in the second chord, and again the user can choose which of these, V, will form the LIP.

7. Where only one given chord is involved, the procedure must be modified. Essentially the given chord A is duplicated as A' at an additional position k' inserted immediately after k and before $k + 1$. The LIP is here a circular progression between two inversions of the one given chord A. For example, a LIP inserted on a single chord of C would be placed immediately after the given 4-voice C chord and would end with a chord consisting of 2 or 3 notes in the triad of C.

8. The chosen interval V is between two of the four (or three) given voices, and these will be the upper and lower voices of the whole LIP. To be clear, these will not in general be the soprano and bass of the given hymn-tune. The interval V will be the closing interval of a few canonic LIPs, and these LIPs are identified by the program and listed to screen. For example, the 10-10, 5-10, 7-10 and 8-10 patterns in Figure 27 all end in a major or minor 10th; hence these four options together with the circle of fifths in Figure 38h, i will be listed on screen, and one selected at random or by the user.

9. The upper and lower voices of the selected LIP are entered into a dummy array D. The first entry to be made is the right-most chord, where the interval V is precisely that of the chord B (or A'). From there D is filled leftwards from its right-most index (15) towards index 0. The canonic LIPs are of two types: those which have only one chord per cycle (e.g 6-6-6-..) and those with two (e.g. 7-6-7-6-7-6). For a 1 chord/cycle LIP the upper and lower voices are added to array D by a formulae of the type $U(final - j) = U(final - j + 1) + 1$ where U is the pitch of the upper voice on the diatonic scale and j indexes the position in the dummy array D. For 2 chords/cycle LIP there are alternating formulae depending on whether j is even or odd. Upper and lower notes of each pair are added to D until at least two have been added and until the two notes both lie in the triad of the first given chord A. At this stage the dummy array D contains a sufficient length of the chosen LIP to bridge from A to B (or to A'). The two notes of its first chord lie in A and the notes of the last in B (or A'). However the section inserted between A and B is normally only the parts lying between but excluding the first and last chords of D. That is to say, where A and B are distinct, the chords of the LIP which are musically equal to A and B are not normally inserted, since to do so would produce two adjacent A chords and two adjacent B in the final piece.

10. Where the LIP has three notes per chord, one of the unused voices is 'borrowed' for the duration of array D. Otherwise the unused voices are allocated a code which denotes a rest.

11. The next stages are directed towards allocating a duration to each chord in array D such that D can be inserted between A and B to join smoothly and fit with the time signature. As an example, suppose the time signature is triple – 3 crotchets per bar. A might be a dotted minim on beat 1, a minim on beats 1 or 2, or a crotchet on beats 1, 2 or 3. B will have a duration consistent with A's and with the 3 beats per bar. (I do not allow the given melody to have chords tied over from the previous bar.) When the LIP has only 1 chord/cycle, this must as far as possible be inserted in chords of equal length. In the simplest case, if A is a dotted minim starting on beat 1 so that B also starts on beat 1, the LIP accumulated in dummy array D will be inserted at beat 1 of the first inserted bar. The duration of each note could be a dotted minim such that each chord of the LIP fills a whole bar.

12. However inserting all of array D at one bar per chord can make the LIP last too long if there are more than about 4 chords to be inserted. For longer inserts minims or crotchets might be better. The challenge then, however, is to arrange them to fit with the time signature. Clearly in triple time 3, 6 or 9 crotchets will readily fit into 1, 2 or 3 inserted bars, but what if there are 4 or 5 chords? I have dealt with these awkward numbers using devices such as:

    - if there are 5 chords (or 2 or 8) the number inserted is increased by 1 to make a multiple of 3, the time signature. This is done by including the first chord in D, which has the notes of triad A. Note that where there is only one given chord A involved, the last chord of D is always included since it is necessary to repeat the triad of A' in the final piece.
    - if there are 4 chords and A and B are distinct, the number inserted can be increased to 6 by including both the first and last chords of D.
    - LIPs with 2 chords/cycle can be inserted as minim+crotchet per bar in 3 time. The number inserted can if necessary be increased to the next even number by including the A-triad at the start of D.

    There are in fact a large number of cases to be accommodated, and the software coding is correspondingly complicated. The algorithm can cope with pieces in triple and quadruple time, and in the major or minor mode.

13. Once the dummy array D containing both pitch and note durations has been prepared, all entries to the right of given chord A are moved to the right by the necessary number of places, and D is copied into the new positions so created. At the same time flags are set in the virtual 5th voice to mark the new positions of the cadences and the notes belonging to the LIP. Where the LIP is of the circle of fifths type, the flags indicate that certain notes are to be flattened.

14. When a LIP has been inserted, the user is offered the option of saving the file and then of inserted another LIP.

As a detail, the flags set in the virtual 5th voice are made available to subsequent programs which decorates a single melodic line. There they tell Program 9, 10 or 11 that the same rhythmic and melodic motifs are to be applied to sequences which have 2 chords per cycle, as well as to those with only 1 chord/cycle.

10.4 Examples

Figure 44 shows three versions of the 4-part hymn tune of Figure 36 extended, respectively, by 3, 1 and 2 LIP inserts. The original piece is made of 4 blocks of 4 bars with cadences at the end of

Figure 44: Three examples of LIPs inserted into Figure 36 by Program 8.

each block, as marked by the vertical lines between the treble and bass clefs. Program 8 avoids these cadences, and any existing LIPs.

- All three LIP in panel A have only two voices. At $a1$ six notes of a 6-10 LIP in soprano and tenor has been inserted. In the notation of §12.3 the given chord A is the C=I at position $k = 5$, and B the G=V at position $k + 1 = 6$. The notes D and B of chord B fit exactly into the pattern, and indeed the LIP is built in dummy array D starting from this pair. In order for three whole bars to be filled with the LIP, the number of inserted notes has been increased to 6 by including the G and E at the left-most position (lowest index) of D, since these are part of the chord of C major, the A chord. In the bar under the $a1$ label the ear hears C major for 3 beats.

- Only the two given chords at positions 6 and 7 separate the first LIP from the second, $a2$. This has only 4 inserted chords, in soprano and alto in the 10-6 pattern. Again the left-most chord of this LIP has been included to make up 4 inserted chords filling exactly two bars. This is the d minor interval under the $a2$ label.

- At $a3$ two bars of the 5-8 pattern are inserted between given chords $A = 13$, $B = 14$, in soprano and tenor. The last interval of this LIP is the soprano G and tenor G in chord B. The first repeats the harmony of the A chord, again to make an even number of inserts which fit into two whole bars.

- In panel B, b is inserted between $A = 19$ (a minor) and $A' = 19$ also – an example of a single given chord being prolonged. This long LIP is of the 7-10 type and has three notes per chord. It has been constructed around the soprano and alto voices which form the top and bottom notes of each LIP chord. The E and C in the treble clef at position 19 are repeated as the last interval of the LIP immediately before the d minor chord at 20. To accommodate the third, middle note of each inserted chord, the tenor voice has been 'commandeered'. Note how the interval of 7th forms alternately between upper and middle, then middle and bottom notes of this pattern, the 7th always resolving downwards. Therefore in the last chord of the LIP the middle note A of the a minor chord is delayed by suspension of the previous B, and only resolves into the d minor chord at position 20, though in the tenor proper, an octave lower.

- Panel C presents at $c1$ and $c2$ two examples of circle-of-fifths progressions. $c1$ is between A = F major at position 1 and B = G major at 2, an interval of 10th. The notes on which the LIP closes (the first to be created in array D) are the D in alto and B in bass. The soprano has been borrowed temporarily to hold the middle notes. The flags set in the virtual 5th voice are interpreted as flattened notes when the diatonic scale is converted to the MIDI scale just before the output is written to file.

- $c2$ is another example of a single chord A being prolonged, in this case 11. The primary voices are soprano and bass, with the alto being borrowed temporarily for the middle notes. The A chord is repeated as A' at the close of the LIP, and A features again at its start, immediately after 11, in order to make an even number of inserted chords.

As it happens, all the examples in Figure 44 give the LIP a rhythm of minim+crotchet. However, I could show other examples in which the inserted chords occupy one bar each, or only one crotchet each. In quadruple time the rhythm can involve almost any mixture of crotchet, minim, dotted minim and semibreve.

That concludes Part 2 of this account of my computer-generated music programs. We have dealt with Programs 1 to 8, all of which concern, in one way or another, the progression of structural notes and chords of a piece. The next Part looks at decoration of these bare notes and chords.

Part 3: Programs in Group 2: Embellishment

We now turn to the later set of programs (Program 9 to 12) in the suite which elaborate the structures output by Programs 1 to 8. The sections below describe many features of Western classical music, some of which I have endeavoured to code into Programs 9 to 12. My premise is that much music from the 18th and early 19th century can be modelled as having an underlying hymn-like structure which is decorated with surface features. The underlying structure consists of the time signature, the harmonic rhythm, the number and length of phrases, their assembly into musical sentences, the leading between the essential notes of the three or four voices, and any inserted linear intervallic patterns (LIPs). The surface features include

- the shapes of phrases,
- arpeggiation of the harmony notes,
- fine scale rhythmic activity,
- non-essential notes meaning notes which are not part of the harmonic scaffolding or an LIP even though they may be highly significant to the melody.

Before going into technical aspects we look at some simple tunes to understand what makes them work. I judge it essential that the software be able to generate song-like melodies.

11 Melodic interest and phrase structure

11.1 What makes a good tune?

I distinguish between a melodic voice line and a tune. Melodic lines will be singable, but not necessarily interesting or memorable. A tune, in contrast, is memorable and attractive even if quirky. Much romantic music, especially 19th century violin, cello and piano concertos by virtuoso players who had limited ability as composers, are melodic but not notably tuneful, and so are easily forgotten. Some TV adverts (commercials), on the other hand, have catchy tunes to make the produce stick in our minds. Even amongst tunes, however, we know that some are more readily remembered than others, whether they be from the classical repertoire, shows and musicals, film music, favourite hymns, folk songs or pop songs. Why do some readily come to mind, a pleasure to recall, while others seem dull? I ask this because the computer programs should aim to output music that is not tedious, boring or confusing.

The matter is not simple. In a companion study on *www.mathstudio.co.uk* entitled 'What makes a good tune' I selected 60 tunes which have been successful and/or which I consider memorable and effective, and picked out features they have in common. I refer the reader to that article for details. Here are my conclusions, most of which are consistent with general advice found in textbooks and on the internet.

1. All the tunes cited in the study are effective within their own context. A tune is best heard within its intended context. This is particularly true of songs with well know words. If taken too far from its natural place, a song is likely to lose its impact, charm and significance. Context here will mean position and function within a larger work, suite, concert, theatrical or other entertainment. It also implies a texture and instrumentation, as some tunes are song tunes and others best suited to an instrument. Some transpositions and arrangements do not sound well.

2. The basic building materials are notes of the scale of the tonic key and the primary triads I, V and IV, the primary colours of tonal harmony. Triads on vi and ii are used much less frequently. vi appears as a substitute for I at an interrupted cadence, and ii is a preparation for V. The major and minor scales, whether ascending or descending, are strong structures which drive forwards.

3. Many good tunes are in an even number of 2- or 4-bar phrases, a structure which is easily understood and feels rhythmically satisfying. This is particularly true of dance forms. Phrases are often arranged in pairs consisting of an opening one (the antecedent) and an answering one (the consequent). There is a good description of phrases in 'The AB Guide to Music Theory, Part II' by Eric Taylor.

4. Repetition is very important – perhaps the most important device to make a tune memorable. Repetition is effective on all scales:

   - repetition of whole sections notated with repeat signs, giving a binary structure A A B B, or ternary structure A A B A , or rondo structure A B A C A D A,
   - refrains in songs, and echoes of sections in instrumental pieces where the supporting musicians play again the few phrases which the soloist has just stated,
   - repetition of individual phrases either immediately or at various positions along the tune,
   - rhythmic and/or pitch repetition of 2- and 3-note motifs within a phrase,
   - 'kinetic repetition' on a single note, a term coined by the musicologist Arthur Hutchings to refer to a single pitch repeated two or more times in quick succession. Such repetition can add energy and drive the music forwards, and is often used to fit quickly sung words.

5. The rhythm is always clear. Accents are mostly on strong beats of the bar, and there is not a great variety of rhythmic motifs – 3 or 4 motifs usually suffice. More curtails the scope for repetition of those already heard.

6. Almost every phrase will have a recognisable shape. Often this is a rising or falling line, or an upwards or downwards arc. It is better if a phrase does not meander around a centrally pitched note. Long scales, up or down, can be effective as they have a strong sense of direction with attendant emotional pull.

7. A pitch range of about one octave. Few tunes span less than a 6th and few more than an octave+5th.

8. Pitch motion from note to note is mostly by step (diatonic interval of 1 or 2 semitones), with some leaps through 3rd, and fewer leaps through 4th, 5th or 6th. Larger leaps are best placed on accented beats where their effect is strongest. Sparing use of leaps through a octave can be very effective as it is probably the most striking and powerful leap to make.

9. Pitch movement and repetition must be matched to the required character of the music. Where a lively tune is wanted, kinetic repetition on one note is useful, and so is arpeggiation of a primary triad. Stately tunes will generally have more stepwise movement, with the upwards and downwards leaps carefully placed on accented notes only, to give an emotional impact of, say, uplift or solemnity.

10. Establishing tonality from the first two or three notes. Of the 58 diatonic tunes examined in the companion article, 18 (33%) have an anacrusis, and in most cases this is scale degrees 5 → 1, dominant to tonic. In 29 (50%) of the tunes, the note on the first beat of the bar is 1, the

tonic: in 18 (31%) it is the 5th, and in 11 (19%) the 3rd. That accounts for all first notes after any anacrusis. In other words, the first accented note of every tune is a note of the tonic triad I, with scale degree 1, 3 or 5. The movement from first accented note to second accented note is most commonly the upwards jump 1 → 3, followed in frequency by the downwards 5 → 1. This means that the most common notes at the start of a tune are scale degrees 5 - 1 - 3 in that order, with the 1 on the first beat of the first full bar.

11. The less frequent accented notes at the start of a tune are the steps 1 → 2, 1 → 7, 5 → 6 and the downwards intervals 5 → 3 and 3 → 1. Intervals such as 1 → 4, 1 → 6 appear to be even more uncommon, though the popular hymn tune Crimond starts 1 → 6.

12. Surprise can be useful to make the tune memorable – something that convention would not expect. This might be a dotted or double dotted rhythm, a syncopation, a local modulation to the dominant, or a large swerve in key up by a minor 3rd as in Peter's theme from 'Peter and the Wolf' by Prokofiev.

11.2 The shape and size of phrases

This subsection presents some statistics on the shape of phrases – the contour of their voice line – and on the musical intervals which span phrases. The length of phrases in songs is determined by the need for the singers to take a breath. It has already been established in §7.2 that typically a phrase lasts 2 bars, whether it be in 2- 3- or 4-beats to the bar. In each phrase the harmonic background will contain between 2 and 4 triads, the average being $3\frac{1}{3}$. The harmony could be just I - V or Ia - V - Ib, or I - IV - I - V, with the chords spun out in length to two bars by arpeggiation, passing notes, auxiliary notes, octave leaps, and other surface devices. Phrases are usually paired and arranged into groups of 4, 6, 8 or 10 to make a musical section. The length of sections in 18th century dance forms was determined by the patterns of steps and hand holds the couples would take before resting momentarily in their places. Each musical sentence of 2 to 4 bars comes to a rest in some type of cadence, and Program 6 identifies these and allocates them to bars suitably spaced throughout the piece. In my companion article on 'What makes a good tunes?' at *www.mathstudio.co.uk*, referred to above, there are ample illustrations of a tune having typically 4, 6 or 8 phrases, some repeated, each of which generally has a fairly well-defined shape, and which together fit into a structure of typically 8, 12, 16 or 20 bars, such as one half of a movement in binary form, which itself has a shape.

Because these structures are normal, the program must try to emulate it. In so far as guide curves are used to control the choice of notes in each voice, they therefore need to be designed on at least two length scales: the phase and the section. The local phrase shape is probably the more noticeable and hence important, but succeeding phrases cannot have arbitrary shapes; there must be balance with one phrase mimicking or complementing another. Furthermore, this means that the length of music which can be concocted with Programs 1, 5, 6 and 7 cannot be longer that about 20 or 24 bars. Longer pieces must be stitched together from two or more such sections. Each section could have a different key centre, attained through modulation. This means that the programs are not well suited to producing fugues, for which many melodic devices of imitation and seamless counterpoint are essential. This section presents further statistical information on phrases to guide the software development. The actual filling out of block chords into melodic shapes is done by breaking the chord into its notes in an arpeggio, and by adding harmonically non-essential notes. Types of non-essential notes include:

- passing notes, which fill in the intervals between one harmony note and the next in sequence in the same voice. Inserting passing notes will convert an arpeggio into a scale in notes of shorter duration.

- auxiliary notes, also called neighbouring notes. These differ in pitch from the parent note by ±1 degrees of the prevailing scale: that is by one or two semitones.
- suspension, which is a note of the previous harmony chord which is maintained in sound while the chord changes to the next.
- appoggiatura, which is an accented 'wrong note' usually one scale degree from the harmony note which is has usurped, and which then resolves to that harmony note.

Before we examine the form of individual phrases, it is worth seeing examples of how phrases are fitted into longer sentences and musical sections. Consider Figure 45 which is the first part of 'The Holy Boy' piano piece by John Ireland. The phrases are clear and singable, and it is also clear how they fit into a larger structure. The section had a double arch structure ∩∩ and the second arch rises higher than the first up to top G, held for one beat to give it emphasis. The listener hears 8 simple 2-bar phrases which grow then fall, grow again by repeated phrases and reach a climax at top G, after which they fall a long way to bottom D. The two-bar tail is to allow continuation of the piece. The phrases structure is $abcd|abef$. All phrases have the same rhythm apart from d, which has a dotted note. In terms of the guide curves introduced at Program 4, each phrase could be controlled by a local guide curve, and all these local ones controlled in turn by a global guide curve which gives the double arc shape to the overall voice line.

The second extract is from a children's piano piece by Gerald Briscoe. Again we see the rise and fall of each phrase, then three short phrases in sequence, rising up the triad of G major and closing in a cadence. The voice line has a single ∩ shape. The third extract is from a Chopin nocturne. The global shape is an upwards arch ∩ with the four phrases fitting neatly within it.

To examine the shapes and 'height' which phrases commonly have, I have determined some salient properties of 132 phrases taken from 37 different works, many of which are in 'What makes a good tune'. For each I noted

- its shape,
- the interval between the highest note and the lowest,

Figure 45: Three extracts showing how individual phrases are made to fit together within an overall global structure with its own shape. Phrases are marked by the red curves.

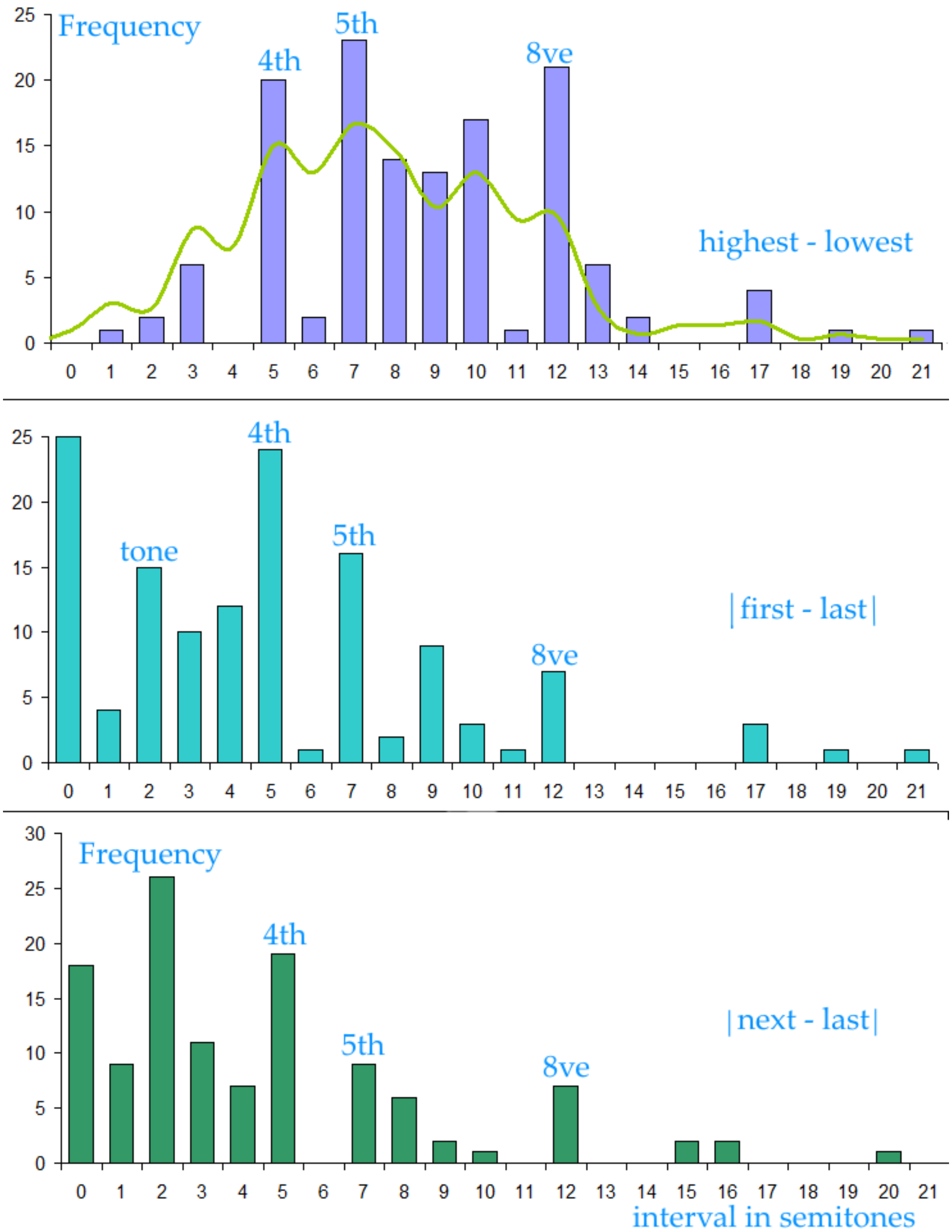

Figure 46: Statistics on the frequency of occurrence of various intervals in a sample of 134 phrases. Top panel: interval between highest and lowest note in phrase. Middle: Between first and last note of phrase (absolute value). Bottom : Between last note and first note of next phrase (absolute value). Intervals measured in semitones.

- the interval between first and last note of the phrase
- the interval between the last note of one phrase and the first note of the next,
- whether the phrase has a anacrusis – an upbeat moving weak to strong.

The frequency of occurrence of various intervals is shown in the three histograms in Figure 46. The upper panel shows the range from the highest note to the lowest in any one phrase. The wobbly green curve is the running mean. The median is 8 semitones (minor 6th), and the three most common values are perfect 4th (5 semitones), perfect 5th (7) and octave (12). Perhaps surprising is the range of 10 semitones, a minor 7th. Few phrases have an excursion less than a 4th or greater that an octave, though just a few have octave+semitone (13), or octave+4th (17).

There are about as many upwards intervals between the first and last note of a phrase as downward ones, so I have combined these by recording the absolute value $|\text{first} - \text{last}|$ in the central panel of Figure 46. The most common value is 0, meaning that the phrase ends on the note it started with. This is perhaps surprising. A significant number of phrases change by only a major 2nd (2 semitones), though the most common interval is a perfect 4th. Also common is 5th, with major and minor 3rds, major 6th and octave also fairly frequent.

The bottom histogram looks at the join from one phrase to the next – does the tune continue from the same note or have a large jump? The graph shows a variety of significant intervals spanning to one octave. The most common is a major 2nd, meaning that the next phrase is one scale step away from the last, a result consistent with the predominance of stepwise motion in melodies. Again the most common jumps are by 4th, 5th or octave.

Coming to the shape of phrases, I have recognised 14 fairly well distinguished contours. I have labelled them $+1$ to $+7$ and their mirror images, -1 to -7. They are sketched in Figure 47 and an example given of each plus their relative frequencies within the 123 examples. They are by no means equally common; the most frequent are 7− (18%), 7+ (14%), 5+ (15%), 6− (14%) and 4+ (12%). Much less common but still significant are the zig-zags 3+, 3− and the falling scale 2− (7%). I noted only 20% of phrases having a weak up-beat (anacrusis). Most phrases start on a strong beat.

Figure 48 gives several more examples of phrases, and the reader is invited to identity their type using the labelling of Figure 47. Some comments are:

- The Bach example is the whole first section of a binary gavotte. It has four beautifully balanced phrases, the first three of which share the same 3-crotchet arpeggio motif which opens the first phrase. The C♯ implies a move to the dominant triad on which the section closes. The first three phrases have a snaking descending shape close to type 7− though 3− is also similar. The last is an upwards arc, ∩, type 4+.
- Schumann's first 2-bar phrase rises in a type 2+ shape through the arpeggio of F major, then falls in the second phrase through a cascade of zig-zag 3+ shapes to the dominant triad, C major. These work together into a 4-bar sentence. There is imitation of phrase 1 at phrase 3, which rises even higher, when the line falls in a gentle line to the dominant note with a type 6− curve. The whole line has two matching arcs, the second reaching higher than the first to give the line itself the background shape of one long arc.
- The first phrase of the Handel prelude is a 3-level terrace arpeggiating the tonic triad of G. The four short following phrases are in imitation and paired. It closes with a descending scale.
- After the anacrusis, the first phrase of the Schubert impromptu is essentially level, with just enough range to arpeggiate the tonic triad. The second phrase is similarly horizontal in the middle and hits its highest note only once, at the end.
- The Mozart fantasie opens with three strong arcs, ∩, the first two being almost in sequence. The last two phrases quoted here are strongly falling arpeggios, \, type 2−..
- The Mendelssohn song without words has two phrases with similar rhythms and complementary $4+, 4-$ shapes.

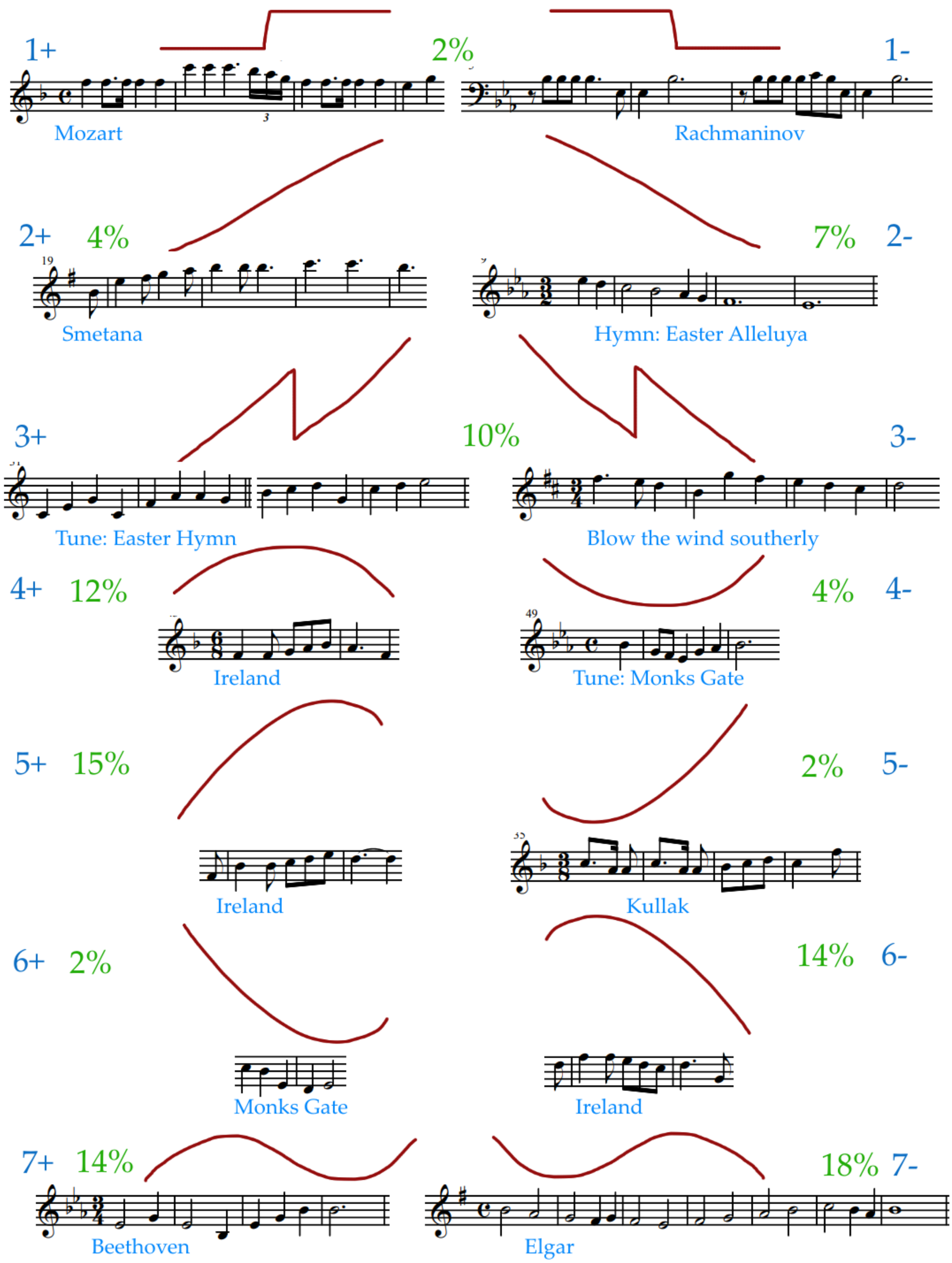

Figure 47: Types of shape of single phrases. The % figures are their relative frequencies amongst the 132 examples examined. An example of each is quoted.

- The first Chopin example shows two balanced 4-bar upwards arcs, types 4+, 5+. The highest note occurs only once, half way through the first phrase.
- Chopin 2 (the Raindrop prelude) has a lazy-S shape, ∽, type 7−.
- Chopin 3, a waltz, also has two balanced type 6− arcs, the second higher than the first.
- Chopin 4, a nocturne, has two wide leaps. The last phrase is rhythmically quite different from the previous three, and has the transposition upwards through an octave – a zig-zag 3− type inserted into what otherwise would be a simple falling type 2− scale,
- The final Chopin example, from the *lento* B minor prelude, has three strong ∩ type 4+ arcs,

Figure 48: More examples of phrases showing a range of shapes and peak-trough intervals. Note the opening and answering phrases in almost all examples; they often have similar rhythms.

closing with a descending type 2− scale/arpeggio, which become overlaid by a counterpoint above, as shown. Each arc rises higher than the previous, a practice already noted.

These examples show how the overall structure may rise then fall, as if controlled by a long, overarching guide curve. Individual phrases within this global shape are shaped in pitch and rhythm to repeat or complement previously heard ones, and work together, often in antecedent-consequent pairs, to form a short musical sentence. Where there are four phrases, the fourth is sometimes significantly different in rhythm from the previous three. There is order within variety, and variety within order – an essential aspect of classical music.

11.3 Rhythmic motifs and patterns

Each phrase will have a rhythm. so we now examine the rhythmic patterns in some 18th century music. Samples of short pieces from that time show that it is common to have the same few short rhythmic patterns recurring throughout the movement. Each such memorable pattern lasts typically between one and two bars. Some are repeated in musical sequences. The listener soon becomes familiar with these patterns and their repetition does much to give the piece its integrity and character. These characteristic patterns are usually built from much shorter elementary sub-motifs, each lasting typically only one beat.

To be clear, look at Figure 49 which is the full right hand part of the short keyboard sonata in G, K63 by Scarlatti. Ignoring the pitch and phrasing of the notes, there are 12 different rhythmic patterns in its 60 bars. In Figure 50, at the bottom line c), I have collected five of these bar-long rhythms. Pattern 12 occurs only once, so on average the others each operate over $59/11 = 5 \cdot 4$ bars. Each minim-long pattern is clearly built from a small number of sub-motifs. Figure 50a shows two ways used to divide a minim, and b) shows four ways for a crotchet. As there are 2 crotchets in a bar and 5 sub-patterns in b), each half bar could be filled from 5 possible permutations, giving $5^2 = 25$ different bar-long rhythmic patterns. In order to maintain integrity Scarlatti has used only 11 = 44% of these. If there had been 3 beats per bar and 5 sub-motifs, the possible number would be $5^3 = 125$. The number of patterns lasting a minim is further increased by dotting quavers and semiquavers and by introducing triplets, ties and rests.

So how much rhythmic variation and complexity is there in music? Every composer must have in mind a balance between having sufficient patterns to maintain interest, yet keeping the number small enough to ensure unity of the piece. To gain some semi-quantitative data on 18th century practice I have counted the number of different rhythmic patterns in 24 pieces, and noted the elementary sub-motifs from which they are built. I have had to decide what duration would be taken by a distinctive rhythmic pattern. In most cases this is the whole bar, but in a few it is only half a bar. As might be expected, the statistics depend on the style and tempo. The general trend is for the slower and more thoughtful movements to have more rhythmic complexity – a different pattern roughly every 2 or 3 bars. It is convenient for our purposes to quote the average number of bars (or half bars) before the pattern changes. This, as I have just said, is 2 or 3 for dignified *adagios*. For muscular *allegro* movements there might be between 6 and 10 bars per pattern, whilst light, fast, dance-like movements have even fewer patterns – typically 14 or more bars per pattern. Two related observations are that triplets are not often combined with quavers, semiquavers and other divisions into 2, and that cadences generally have at least one long note.

My evidence is collected in Table 12. This lists 28 movements in a variety of styles in order of increasing average number of bars/half bars per rhythmic pattern. To explain the columns,‘mvmt’ is the movement number in a work of several movements. ‘Unit’ lists what I have taken to be the

Figure 49: The right hand part of Scarlatti's sonata in G, K63, marked with its 12 rhythmic patterns. From an ABRSM publication edited by Goldsbrough.

duration of each rhythmic pattern. 'Length' lists the number of such units in the movement. For an example take the second row, the graceful introductory movement of the violin sonata in F attributed to Handel from which Figure 18 is a quotation. I count 25 different patterns in its 54 bars of slow 3 4 time. Taking a whole bar as the pulse, there are on average only $2 \cdot 2$ bars per pattern; it is rich in rhythmic variety which contributes to its sinuous, almost rhapsodic melodic line. Looking at the elementary sub-patterns, Handel has divided a dotted minim only 4 ways, and a crotchet only 5, yet with 3 crotchets per bar (*i.e.* per pattern) he has used only 25, or 20%, of the 125 possible permutations. In contrast to the grand slow introductory movements of many 18th century sonatas, the last movements are often gigues which have very little rhythmic variation. We may assume that the composer judged that any significant changes would disturb the joyful flow of these dance

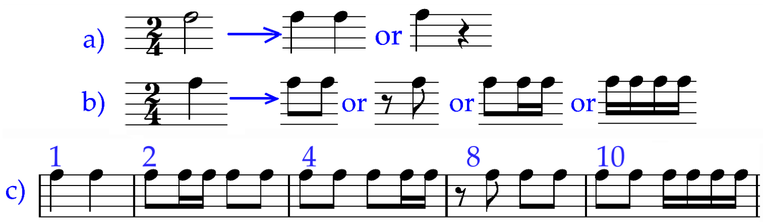

Figure 50: Construction of rhythmic patterns. a) division of minim, b) of crotchet, c) assembly into a five bar-log patterns, labelled as in the Scarlatti sonata.

Composer	work	mvmt	tempo	time sig.	unit	length	no. of patterns	units/ pattern
Handel	Vln sonata in g	1	Larghetto	4 4	minim	34	18	1.9
Handel	Vln sonata in F	1	Adagio	3 4	dt minim	54	25	2.2
Handel	Vln sonata in D	1	Affettuoso	4 4	minim	52	24	2.2
Handel	Vln sonata in F	3	Largo	3 2	dt minim	11	5	2.2
Handel	Vln sonata in E	1	Adagio	4 4	minim	38	16	2.4
Hook	Sonatina 2 E♭	1	Allegro mod	2 4	minim	24	7	3.4
Hook	Sonatina 12 E	1	Andantino	4 4	minim	46	13	3.5
Kuhlau	Sonatina Op 55 No 4 F	1	Allegro mod	3 4	dt minim	44	12	3.7
Hook	Sonatina 4 A	3	Minuetto	3 8	dot crt	32	7	4.6
Bach	Cello Suite 3	5	Bouree 1	2 2	semibreve	28	6	4.7
Handel	Vln sonata in D	4	Allegro	3 4	dt minim	72	15	4.8
Scarlatti	Sonata K471		Minuet	3 4	dt crt	42	8	5.3
Marcello	Cello sonata in e	1	Adagio	4 4	crt	58	11	5.3
Hook	Sonatina Op 12 No 1 D	1	Allegro spirito	2 4	minim	52	9	5.8
Scarlatti	Sonata K63		Allegro	2 4	minim	59	10	5.9
Hook	Sonatina 12 E	2	Rondo allegro	2 4	minim	54	9	6.0
Handel	Vln sonata in E	4	Allegro	3 8	dt crt	67	10	6.7
Corelli	Vln sonata Op 5 No 7 in d	4	Giga Allegro	6 8	dt minim	69	9	7.7
Handel	Vln sonata in F	2	Allegro	4 4	minim	88	10	8.8
Marcello	Cello sonata in e	2	Allegro	4 4	minim	58	6	9.7
Handel	Vln sonata in F	3	Largo	3 2	minim	44	4	11.0
Handel	Vln sonata in g	4	Allegro	12 8	dt minim	60	5	12.0
Corelli	Vln sonata Op 5 No 7 in d	1	Courante	3 4	dt crt	74	6	12.3
Mozart	Vln sonata K304	1	Allegro	2 2	semibreve	205	16	12.8
Beethoven	Piano sonata Op 2 No2 in A	1	Allegro viv.	2 4	minim	118	9	13.1
Marcello	Cello sonata in e	4	Allegretto	3 4	dt minim	42	3	14.0
Bach	Cello Suite 3	7	Gigue	3 8	dt crt	108	7	15.4
Corelli	Vln sonata Op 5 No 7 in d	4*	Giga Allegro	6 8	dt minim	50	3	16.7
Handel	Vln sonata in F	4	Allegro	12 8	dt minim	104	4	26.0
Marcello	Cello sonata in e	4*	Allegretto	3 4	dt minim	37	1	37.0

Table 12: The average number of pulses per different rhythmic pattern in some 18th century pieces. Vln = violin, dt = dotted, crt = crotchet. * = passages in a movement with very few patterns.

movements, as if the dancers had stumbled or tripped up. As the two passages marked * in Table 12 attest, such works can go on for many bars in steady flowing triplets, quavers or semiquavers. The same can be said of many of the movements in Bach's solo cello suites and solo violin partitas such as the preludes, allemandes and courantes. In contrast the sonatina movements by Hook and Kuhlau have a relatively large number of patterns for their short duration.

Viewed overall, Table 12 shows that the average number of bars per rhythmic pattern is a determinant of musical style. The implications of the above for an algorithm are that the user must first specify the style of the piece, particularly its pace and mood. That together with the time signature and the overall length will determine the number of different rhythmic patterns. These in turn can be built from a few elementary sub-motifs, perhaps using some rests and ties.

12 Further characteristics of classical music

12.1 Motifs and compound melody

In writing for instruments Bach, Handel and their contemporaries used small groups of notes to decorate and thereby substitute for intervals between voices in harmony. We have already seen examples in my elaboration of a chord sequence into a violin study Figure 5 in §4.1 and in the Handel extract Figure 18. A flowing texture, often weaving amongst two or three parts, was common. Block chords were used sparingly. Figure 51 A shows methods whereby a single note was divided rhythmically by using neighbouring notes. My reading of the scores, however, is that division was usually done with regard to harmonic intervals, triads and fuller chords. In Figure 51 B are about a dozen examples from the 18th century showing how chords of two, three or more notes have been broken up. These patterns were adapted to 2-, 3- and 4-in-a-bar and compound times, and were characteristic of keyboard and solo instrumental music of that time. Corelli's violin variations on La Follia, Op 5 No 12, are almost a catalogue of the possible ways to decorate a triad.

Figure 51: Motifs common in the 18th century for elaboration of A: a single note, B: an interval of 2 notes or a triad. The notes being elaborated are shown as a chord at the beginning of each example.

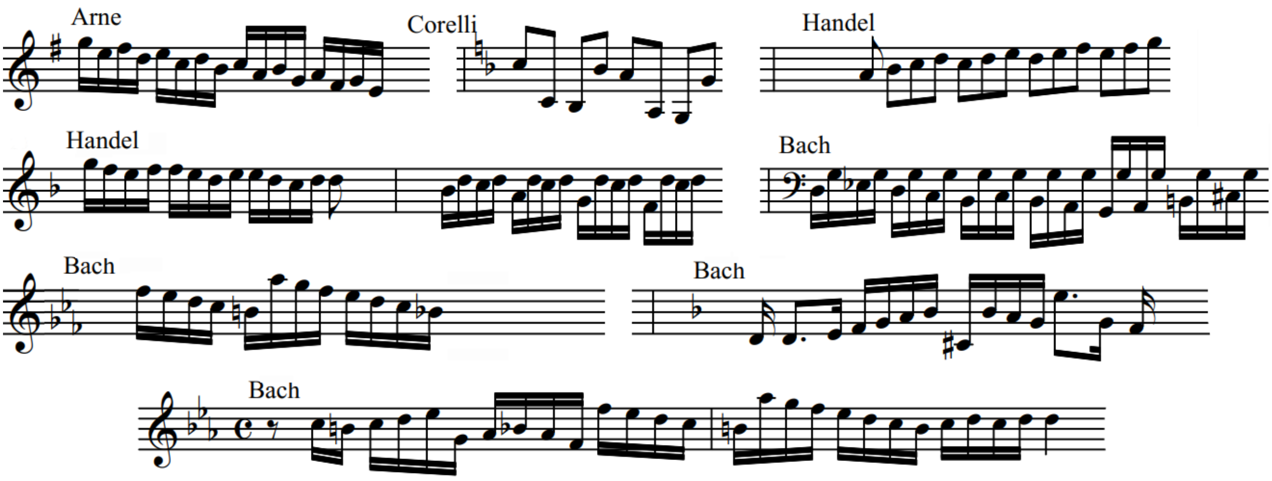

Figure 52: Elaboration of scale passages, including with a pedal note or octave jump.

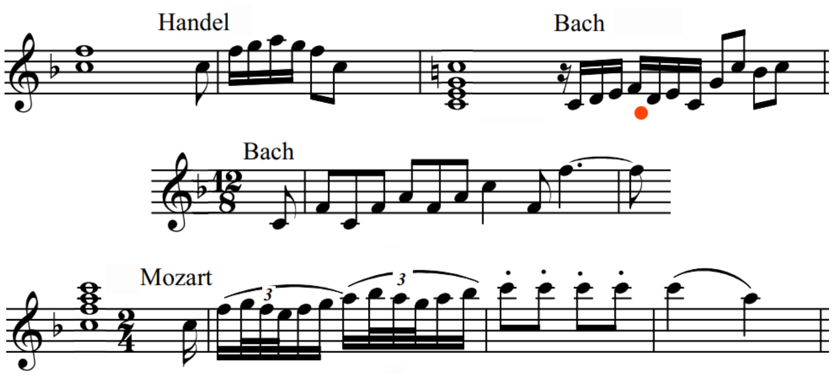

Figure 53: Memorable themes formed by elaboration of a triad.

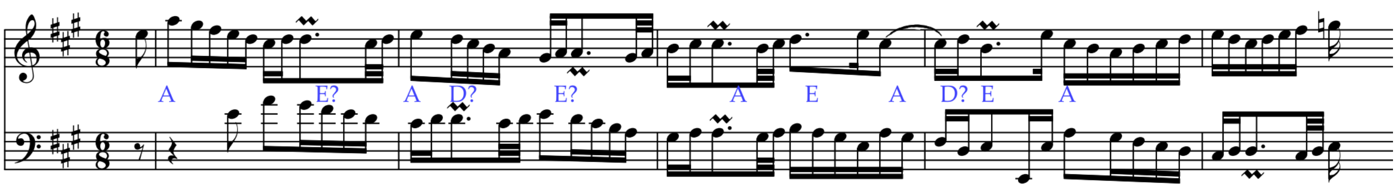

Figure 54: Bach's imitative gigue, English Suite No 1 in A.

Another notable characteristic is the treatment of scale passages, where the scale may be broken by an octave jump, or decorated with escaped notes, or tied to a repeated pedal. Examples are given in Figure 52. Jumps through a large interval can give the listener the impression that a second voice line is present, so the impression is given that an instrument which can sound only one note at once, such as flute, is playing two voices. Real invention is shown where the triads have been given such rhythmic identity that they stand as themes in dances, fugues, arias and other extended pieces. Figure 53 gives examples from Handel, Bach and Mozart.

One side effect of decorating triads by passing and neighbouring notes is that other harmonies become fleetingly hinted at. We hear this in Bach's C major Two-Part Invention where G7th is implied within a C major triad, marked with the red spot in Figure 53. Two-part writing often has an ambiguity as to the instantaneous triad we are hearing, one of whose notes is missing. An elaborate example is the opening bars of the gigue from Bach's first English suite in A, Figure 54, where canonic imitation by the lower part produces flashes of E 7th and D major in what is essentially just an A major chord extended over 5 bars. If the passing note is given the duration of a beat, the harmony so formed is heard as a distinct chord. Recall from §6.2 that textbooks explain how the dominant 7th and other 7th chords were traditionally generated by falling passing notes. Similarly the cadential 6 4 is produced by the dominant note held in the bass while the upper voices pass through the other notes of tonic chord on their way to the dominant: V7 - Ic - V.

The large leaps, as in the scale examples in Figure 52, were sometimes continued for several bars to give the sound of two or even three interlaced voices in counterpoint. The term 'compound melody' is given to this sustained impression. It is a particular characteristic of Bach's instrumental writing, by which he wrote contrapuntal works for solo violin and 'cello. Figure 55 quotes three instances from the 'cello suites. Voice leading is respected in each part of these compound melodies; observe the descending scales from bar to bar.

Figure 55: Compound melody in Bach's cello suites. A: Suite 1, allemande, B: Suite 1, minuet 2, C: Suite 3, boureé 1.

12.2 Some stylistic features of the 18th century

Here I note some common features of 18th century instrumental music, particularly keyboard music, which occur alongside the phrase, sequences and motif patterns described above.

Many music lovers will regard fugue as the archetypical musical structure of the early 18th century, with Bach's '48' and his organ fugues the quintessence of this style of writing. I do not propose to discuss this at any length, partly because it soon becomes very technical, and partly because I have not yet attempted to write a program which will create a fugue subject and an effective answer. Imitation occurs in many pieces other than fugue, and arises when the same decorative patterns in rhythm and pitch are superimposed on a harmonic framework.

In §3 I presented statistics on the frequencies of triads and their progressions, showing that I - V and V - I are by far the most common. We can look at this in a slightly different way and say that in any sequence of triads, denoted by Roman numerals, it is feasible to extend any I triad by splitting it into two and inserting another triad, so that I → I - X - I. X is called a 'passing chord' and could be V, IV or sometimes ii. Similarly V can be extended by splitting it and inserting I to form V - I - V. V - ii - V is also musical, being the counterpart of I - V - I in the dominant key. The Ia - V - Ib subsequence is common and often has V in 2nd inversion as Ic, a 'passing 6 4 ' chord. Triads Va and Ic have the same bass and function harmonically as the same dominant chord. In Figure 56 is a wonderfully decorated example from a Haydn sonata in F major. The suspended F in bar 2 creates a Ic chord before V itself is formed on beat 3. Suspensions are described below.

The sequence I - IV - I is not uncommon in music of the period. I quote in Figure 57a, b a few bars from two charming well known minuets which Bach wrote for little Anna Magdalena. In the first (top panel) beat 3 of bar 1 gives a fleeting sound of A minor, but this is not harmonically significant; the first two bars are an arpeggiation of the G major triad. In Figure 57c by Mozart the IV chord is in second inversion.

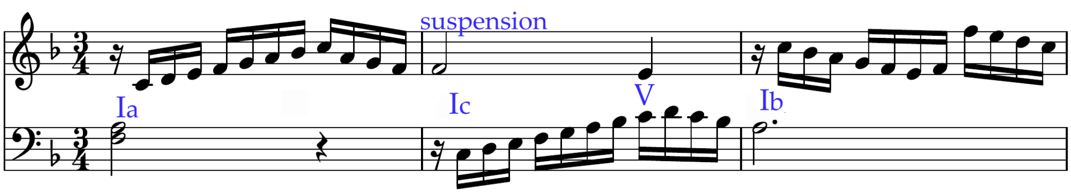

Figure 56: First 3 bars of Haydn's keyboard sonata in F, 1788.

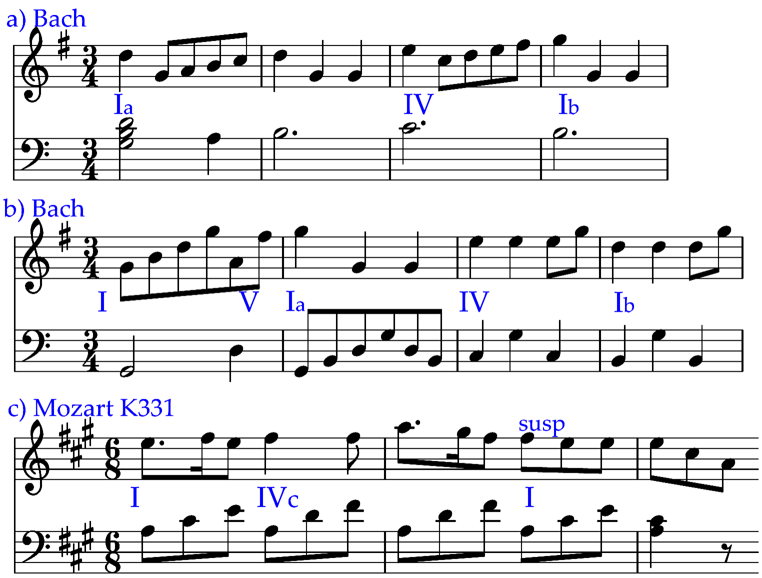

Figure 57: Extracts from two minuets by Bach, and a Mozart piano sonata, all using the I - IV - I subsequence. *susp* indicates a suspension.

The extract in Figure 58 from an 18th century sonata shows several characteristics of 18th century piano writing:

- the I - IV - I subsequence over the first 4 bars,
- the I - V - I subsequence in the last 3 bars,
- the use of second inversions. The B♭ bass note of IVc persists for 4 bars as a sort of pedal note or drone.
- the V7c chord would have figured bass 4 3. Its bass C is a passing note in the short ascending scale from B♭ to D.

A bass pedal note in a rocking left hand is quite common in this keyboard music, as is the fondness for 7th chords. Adding the 7th to a triad makes its second inversion more musically acceptable. Figure 59a, b are examples from keyboard sonatas by Kuhlau and Mozart respectively. Again we see a plain I triad being split and extended by sandwiching in a V or IV chord. A rocking left hand with B forming a bass pedal note is also seen in later music, as in the Liszt extract of Figure 65 of §12.4 below. Yet another characteristic is the fondness for suspensions. The cadential 6 4 is well recognised (Figure 17b of §6.2), but we already have seen it in non-cadence positions in Figures 26g (Bach), 56 (Haydn) and 57c (Mozart). An abundance of suspensions occurs at the beginning of a piano sonata in C by Haydn from 1773, Figure 60.

Figure 58: Opening of a rondo to a piano sonata of 1798 once attributed to Mozart as K498a, but probably by August Müller.

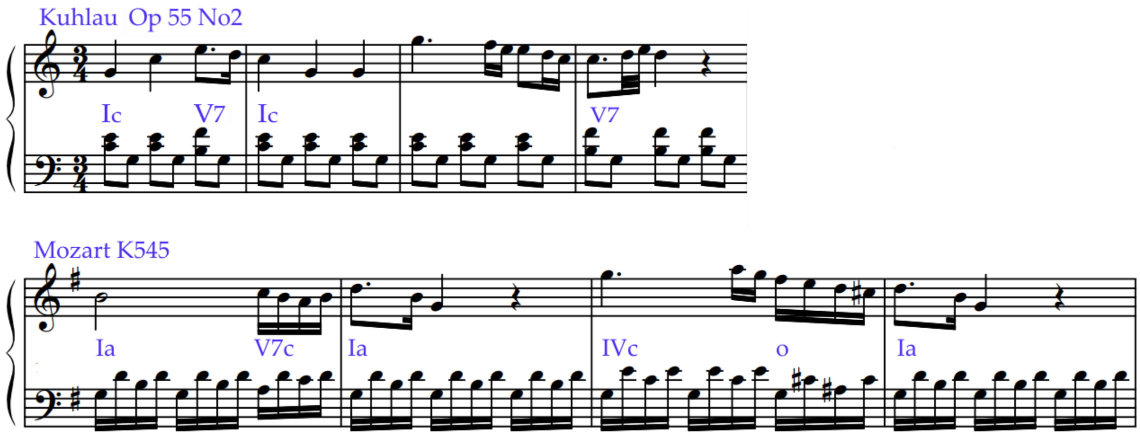

Figure 59: Extracts from two 18th century piano sonatas showing rocking pedal bass, 7th chords and 2nd inversions.

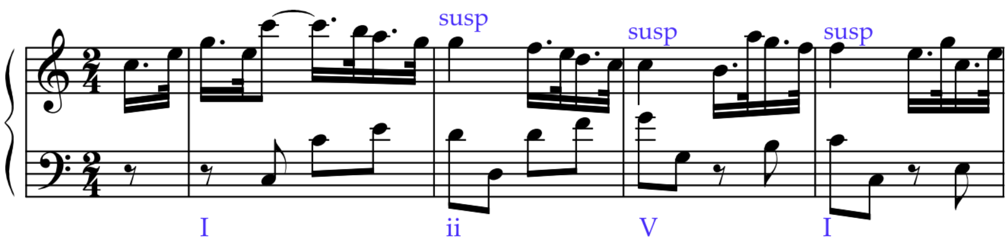

Figure 60: Beginning of a Haydn sonata in C from 1773 showing suspensions in the melody.

12.3 Ornamentation and improvisation

In jazz, which developed from about World War I, there has been much admiration for musicians who could improvise on a given tune or chord sequence. Two centuries earlier, all musicians were expected to be able to improvise, and part of their training was in how to do this within the conventions of the day. It played a central role in baroque and classical continuo playing upon a figured bass, and in the virtuoso display of singers, violinists, keyboard players and other performers. This is especially noticeable in concertos where each cadenza was an opportunity for a renowned soloist to impress his audience, and audiences admired virtuosity in a player perhaps as much as the composition as a whole. This aspect of musicianship largely fell out of classical art music from the mid 19th century. From that time the emphasis in music teaching was more on playing strictly what the composer had written, and in parallel composers wrote more musical symbols into their scores to instruct the performer. In contrast, music manuscripts and printed pieces in the 18th century not infrequently gave only the essential notes, assuming that the performer's training would work on these to fill them out in performance. Ornamentation went beyond mordents, turns and trills, though these were important, particularly when instruments like the harpsichord and forte-piano could not sustain a note. In the mid 1700s the composer Johann Quantz, a pupil of Vivaldi, wrote "In former times the Adagio was written in a plain, dry style, more harmonic than melodic. The composers left it to the performers to make the melody singable, but this could not be well accomplished without considerable additional embellishments ... and since the performance was seldom as the author wished, there has come of this evil some good, namely that composers have for some time past begun to make their Adagios more singable". In emulating music of the 18th century it is necessary to have some regard to how improvised embellishment was carried out. Fortunately we have a number of pieces from the time in which the embellishment has been notated.

Figures 61, 62 and 63 are extracts for three 18th century works where the embellishment is preserved alongside the original notes. The oboe sonata by one of the Marcello brothers, contemporaries of Vivaldi in Venice, has the oboe part in the second, slow movement as written in the upper staff in Figure 61. J. S. Bach transcribed this for keyboard and his version is printed in the lower staff. Modern recordings of the oboe concerto, with string orchestra accompaniment, use Bach's version for the oboe soloist. The four phrases are in sequence over simple harmonies, and at each cycle Bach increases the ornamentation using passing and auxiliary notes.

For other examples we go back two generations to Corelli's violin sonatas, Opus 5. These were first published in Rome in 1700 and shortly afterwards some embellishments in the violin part were written out by Corelli's pupil, Geminiani. Figure 62 shows extracts from three movements of Op 5 No. 9. Observe how Geminiani takes significant liberties, departing in places from the original notes. His method is to elaborate the triad of the harmony at each note, marked in blue. Corelli's sonatas were so popular that they were published several times around Europe. The version published in 1710 by Estienne Roger of Amsterdam has a different style of ornamentation, as shown in the example in Figure 63. Here there are wide roulades, derived from the underlying harmonic triad filled in with passing and auxiliary notes, and played in a succession of quick notes of equal duration. These roulades may not have been played metrically, so the continuo players had to take their tempo from the soloist. Are some of these embellishments over the top? Perhaps. Taken too far, embellishments on the harmonic background can amount to a recomposition of the piece, as in jazz. Even in Corelli's times the German composer Georg Muffat had to warn against the player disfiguring the music with "a disorderly heaping up of barefaced tomfoolery".

The French jazz pianist Jacques Loussier was drawn by the improvisatory character of some of J. S. Bach's music and based his distinguished career with his trio fusing Bach with jazz. We can sense Bach's improvisation in works such as the slow movement of his Italian Concerto, where

Figure 61: Oboe part in the Adagio from the oboe concerto in d minor by Alessandro Marcello Upper staff), with Bach's elaboration for keyboard, BWV 974, (below).

Figure 62: Violin part of Sonata Op 5 No. 9 in A major by Corelli (upper staff) with embellishments by his pupil Geminiani (lower).

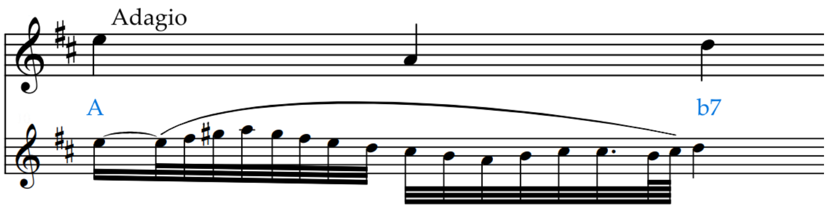

Figure 63: Corelli's simple notes (upper staff) massively embellished by Roger (lower). Op 5 No 1.

Figure 64: Extracts from Bach's keyboard pieces: a) Italian concerto, b) Partitia in c minor.

the left hand of the keyboard treads slow, steady quavers while the right hand weaves a thread of decoration and syncopation (Figure 64a). A work in similar vein is the *andante* which follows the opening sinfonia of Bach's second keyboard Partita in c minor, quoted in Figure 64b. Yet further examples are the opening very slow movements of the sonatas for solo violin in g minor and a minor where the violin decorates single notes and implied chords with arpeggios and roulades.

12.4 Piano music in the 19th century

In the 19th century tastes and outlooks changed to the romantic, nationalistic and picturesque. To express this, music tended to be built from several themes, some song-like, heard over an accompaniment of broken chords. These themes would be 'developed' by modifications and extensions. Composers used less counterpoint of independent voices, but became more adventurous in harmony, through added 7th and 9th notes and diminished intervals, chromatic passing notes, and enharmonic modulations to distant keys. The two examples from Liszt and Franck in Figure 65 illustrate significant departure from the basic diatonic triads. It was a further move away from the 16th century's long vocal lines towards the immediacy of lush harmony – sound heard in the moment rather than spun out in time – and pointed towards those 20th century composers who experimented with all manner of sensory experiences in sound.

Figure 65: Liszt's Consolation and the second theme in Franck's Variations Symphoniques have tunes with chromatic harmony.

As piano building technology and piano technique developed hand in hand throughout the 19th century, so composers developed new ways to present the romantic melodies and harmonies of their piano works. In many pieces the melodies were simple to allow the piano to sing out a song-without-words, while the accompaniments became more full and elaborate, yet needed to be kept in the background. The opening of Schumann's Fantasise Op 17 and Schubert's G flat Impromptu Op 90 are examples. Perhaps these joint trends to simpler melodies, less counterpoint and longer flowing melodies were bound to move together because it is not possible to emphasise one feature of music without simultaneously shading the others. Amateur pianists find it difficult to play this repertoire well: not just because there are so many notes, but because of the challenge of keeping the accompanying notes quiet enough for the tune to sing out, particularly when one hand must play both tune and backing.

Many 19th century piano works were written by concert-giving virtuosi, keen to show off their skills at the keyboard with works of staggering technical difficulty, employing all the notes

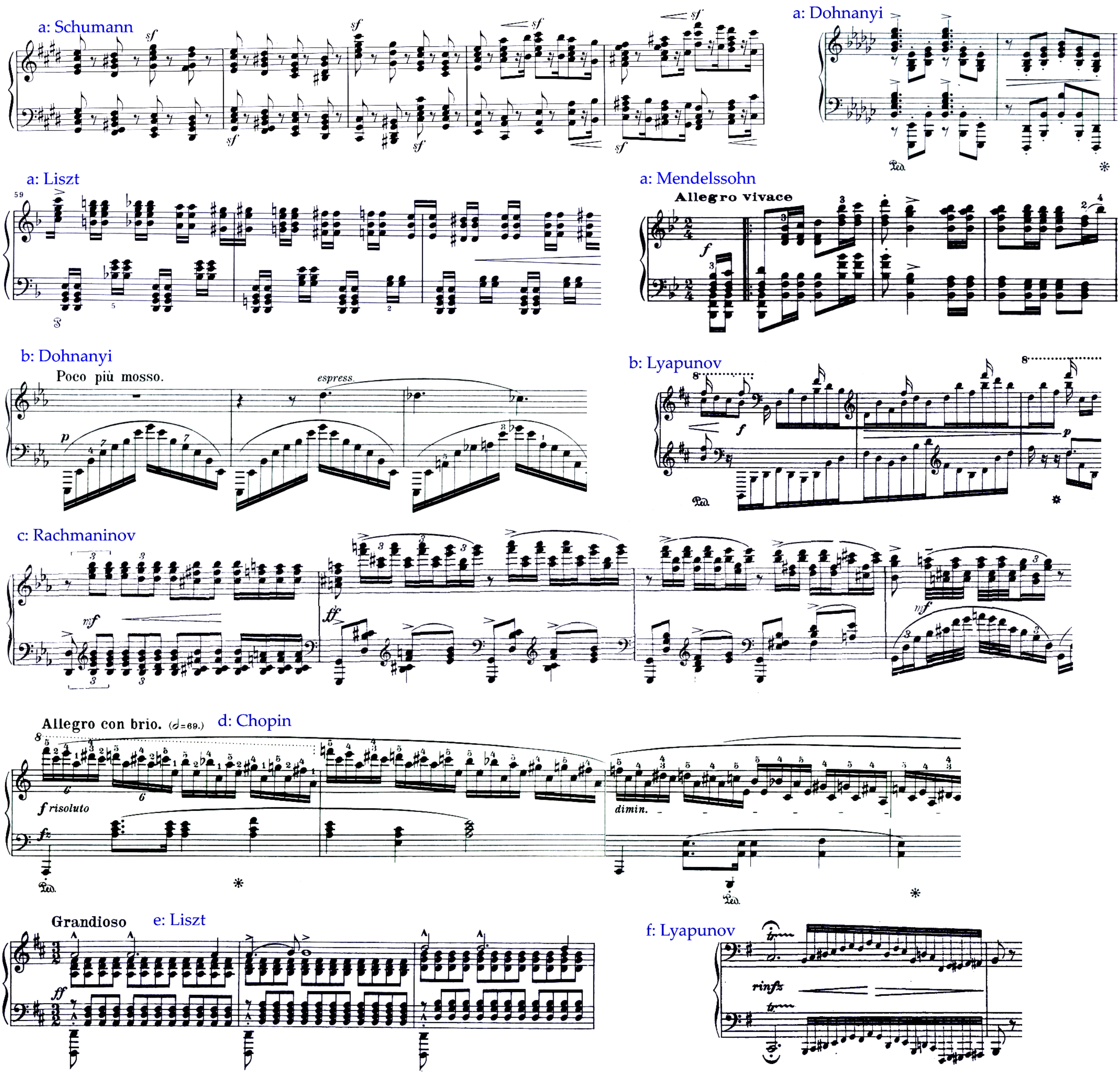

Figure 66: Examples of romantic period piano writing.

which human hands are physically capable of reaching. We find these features of 19th century piano writing which depart far from the basic SATB hymn tune skeleton:

- large block chords are often used, Figure 66a,
- accompanying chords are spread over several octaves, often laid out as arpeggios; Figure 66b,
- textures are often thicker, with more notes sounding together, and piano texture can seem an end in itself; Figure 66c,

- chromatic decorating notes are common, including whole chromatic scales, Figure 66d,
- Mendelssohn, Liszt and others were keen on repeating a chord, in quavers or triplets, to maintain the volume of the piano and create an almost orchestral sound, Figure 66e,
- within both melody and accompaniment the timings may be irregular, with 3, 5, 7 notes to a beat, or 2 or 4 in one hand against 3 in the other,
- pure decoration was occasionally added in the form of roulades – short brilliant cadenza-like passages of rapid scales or scale-like sequences, Figure 66f.

In this welter of notes the voice leading could easily become lost. In contrast with the excesses of the travelling virtuosi, the Lyric Pieces of Grieg are a return to a simpler statement of meaning. Some composers returned to effective counterpoint in some of their piano writing, notably Rachmaninov (e.g. Prelude in D, Op 23 No 4) and Medtner.

The survey of features in music described in §7, §11 and §12 effectively set the agenda for development of this suite of computer programs. The following sections describe my attempts to rise to some of these challenges.

- Program 9 introduces passing notes,
- Program 10 adds rhythmic decoration to a voice line, breaking it into phrases,
- Program 11 arpeggiates chords and so can create accompaniments, and
- Program 12 deals with decoration in which two or more voices are combined into a single compound voice.

I again emphasise that these programs are a toolbox for a musically interested person to use to their taste. This is not artificial intelligence, but computer-assisted human intelligence.

13 Program 9: Passing notes, suspensions and appoggiaturas

Passing notes are ones inserted between two harmony notes to create a smoother, stepwise motion in the voice line. A suspension is a note, usually a harmony note of a triad, which is prolonged to overlap the harmony of the next triad. An *appoggiatura* is an accented non-harmonic note, typically on the first beat of a bar, which resolves by step to a harmony note of the current triad. It may be approached either by step or leap. It takes its name from the Italian word for 'to lean' since it is musically unstable and leans for its resolution into the next note, which must be supported by the harmony. None of these is part of the underlying harmony, but they are important, commonplace devices for adding interest to a melody. Program 9 inserts all of these types into an otherwise sparse vocal line.

I will deal with passing notes first, and restrict to diatonic ones, not chromatic ones (which can bridge the interval of one tone). In principle a section of scale could be inserted into any interval between one note and the following, though probably the intervals most bridged are the major and minor 3rds. Figure 67 illustrates some ways in which this may be done. Cases a, b and c apply in simple time, while d and e are for compound time. There are some subtleties in inserting even one passing note. For instance, the inserted note may fall on a beat of the bar or not. In the former case it is an accented passing note, in the latter a non-accented one. Accented passing notes create a fleeting non-harmonic chord which the non-accented ones do not. The appoggiatura can be regarded

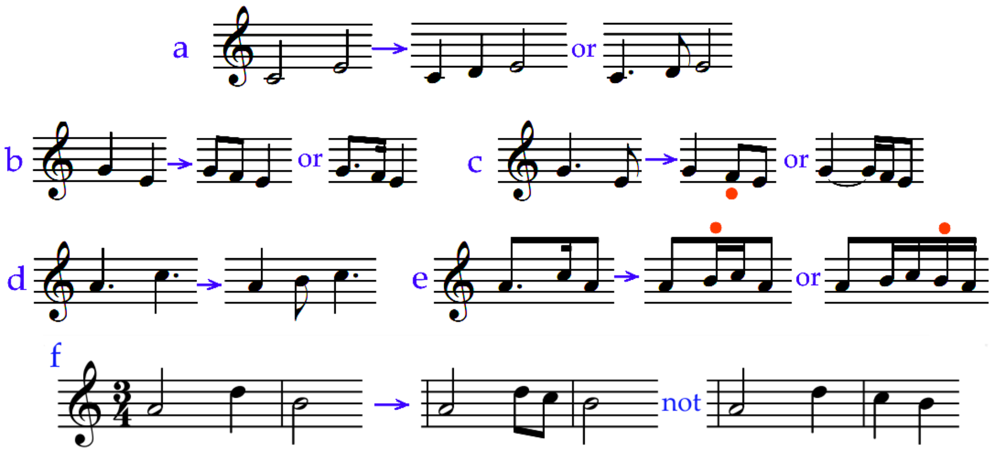

Figure 67: The main forms of passing notes to bridge the interval of 3rd. The red spots indicate accented passing notes.

as an extreme case of an accented passing note, especially if it is approached by step. The cadential 6 4 chord described in §6.2 is created by a parallel pair of appoggiaturas. Any passing note may optionally be part of a non-dotted or a dotted rhythm, or part of a triplet.

Program 9 takes as input a single voice line at a time and inserts passing notes according to a number of options. As in Program 10 and the other decorative program, Program 11, information on the local key is used to adjust the scale from which the passing notes are chosen. This information, along with that on cadences and linear intervallic patterns, is input to Program 9 in an encoded $8T + C$ form output from Programs 5, 7 and 8[14] . The user sets 8 flags which determine

- the minimum duration of a note (usually 3 time units = 1/8 of a crotchet = 1 demisemiquaver),
- whether suspended notes are allowed,
- whether appoggiaturas are allowed,
- whether other accented passing notes are allowed,

[14] For those who wish to know, the triad type $1 \le T \le 30$ in Table-9 notation and cadence positions $C = 1, 2$ are stored for each triad in a single byte using the encoding $8T + C$, a value less than 255. $C = 1$ at the first chord in each cadence pair, and $C = 2$ on the second (closing) chord. Program 8 marks linear intervallic patterns with integers $C = 3$ to 6.

Figure 68: Specimen in 6 8 time showing original at top (a), and at (b) the action of Program 9 in bridging intervals of 3rd, 4th and 5th with passing notes.

- whether dotted rhythms are allowed,
- whether all or any intervals of 3rd should be bridged,
- whether all or any intervals of 4th and 5th should be bridged,
- whether triplets are allowed.

The flag values are 0 = not allowed, 1 = forced wherever possible. 2 causes the computer to choose 0 or 1 at random at each triad to give a 50-50 chance of a suspension, appoggiatura, etc. being inserted. The program will not insert a note into a 3rd if it would be shorter than a demisemiquaver. Also, it will only insert into a 4th or 5th if the original note has at least the duration of one beat, which in compound time is one dotted crotchet, and it will not alter a 4th or 5th if it occurs on the first beat of the piece.

The examples in Figures 68 and 69 illustrate Program 9. In Figure 68 the given voice line in 6 8 time is at 68a, and in 68b this is embellished by passing notes inserted into all 3rds and into

Figure 69: Three elaborations with passing notes of an original (a) using options within Program 9.

most 4ths and 5ths. The given theme is Figure 69a, in 3 4 time, is highly contrived with large leaps for the purpose of showing how the output depends on the option flags. The three examples in b, c, d have different settings as listed in Table 13. Thus in Figure 69b the 3rds have been left unchanged and only those 4ths and 5ths which involve longer notes have been modified. In c) the only intervals bridged are 3rds, and a dotted rhythm has been used. There is an accented passing note on beat 3 of the first bar. In d) the 3rds are bridged in triplets, and there is an appoggiatura on the first beats of bars 2, 3 and 7. Of course, musical taste demands that not every 3rd be bridged, nor every 4th or 5th left unbridged. To add some variation and ease the listener's ear, and to prevent the music becoming too cluttered with decorative notes, Program 9 has the option to set flags in Table 12 to '2' which causes 0 ot 1 to be selected at random.

	Fig 68	Fig 69b	Fig 69c	Fig 69d
Time sig.	6 8	3 4	3 4	3 4
Triplets	0	0	0	1
Appoggiatura	0	0	0	1
Accented	0	0	1	1
Dotted	0	0	1	0
3rd	1	0	1	1
4th & 5th	1	1	0	0

Table 13: Settings of flags used to create the examples in Figures 68 and 69. 1 = set, 0 = unset.

While the passing notes inserted in Figure 68 give a not unmusical effect, those in Figure 69 b to d have the major weakness that the rhythms associated with them are too different and hence not memorable. Controlling the rhythmic and pitch motifs is a significant challenge, addressed in later programs.

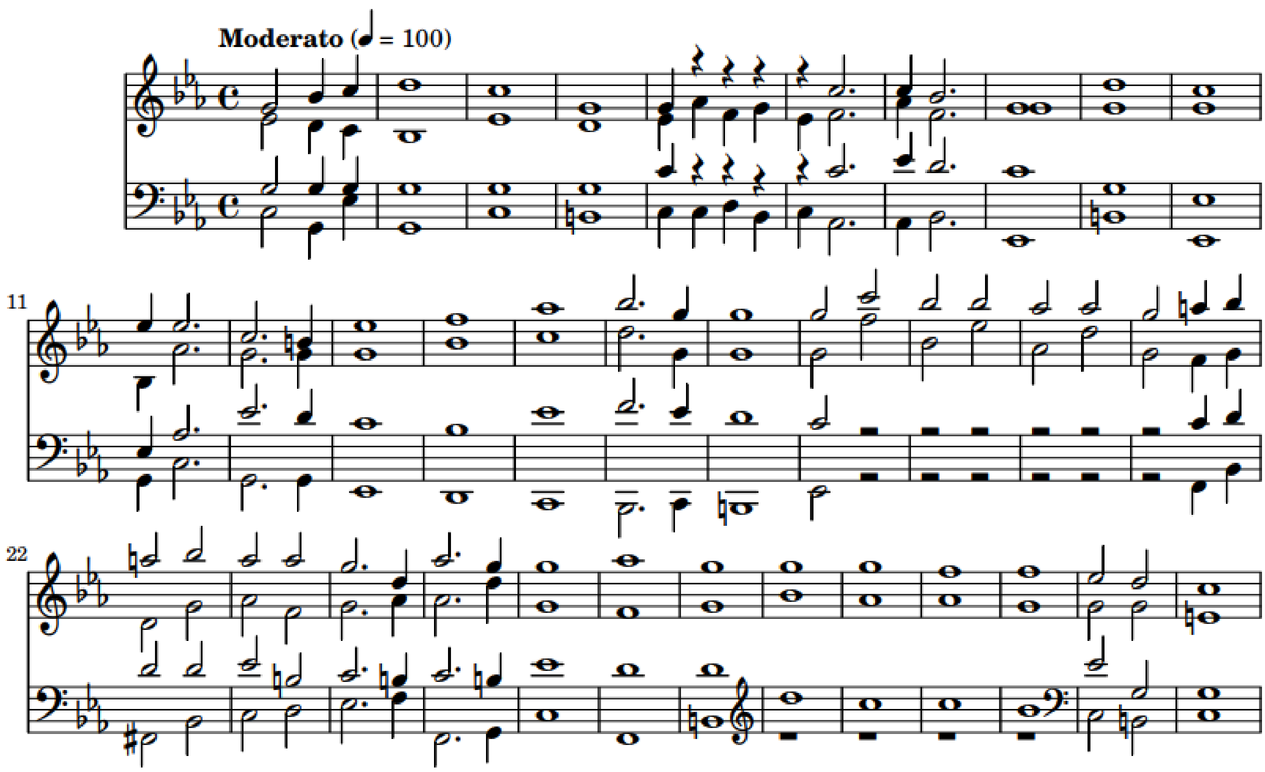

Figure 70: A piece in the minor mode created by Programs 1, 6, 5, 7 and 8. 8 has inserted three 2-cycle LIPs.

The final example of Program 9 illustrates the treatment of linear intervallic patterns and suspensions. Figure 70 is a piece of 34 bars in simple 4 time in the minor mode, created by Programs 1, 5, 6, 7 and 8, and converted to musical notation using Program 3 and Lilypond. I have made a few by-hand changes, mainly to raise the tenor voice by one octave. Using Program 8 I inserted three LIPs starting in bars 5, 18 and 29 – they are obvious from the rests. All are of the 2/cycle type, but involve different pairs of voices. The four voices of this 'hymn tune' were provided as data to Program 9, along with the $8T + C$ code for each chord and the local key of each chord, which was c minor (=13) throughout. Program 9 was then used, one voice at a time, to decorate the piece with passing notes and suspensions. All intervals 3rd, 4th and 5th were forced to be bridged, but the insertion of suspensions, appoggiaturas and other accented passing notes, and dotted notes were all selected by the computer at random. Triplets were not allowed.

Figure 71 shows the elaborated piece created when the four voices are recombined. The LIPs are picked out in red. No by-hand improvement has been carried out. Note these features:

- The passing notes have smoothed the soprano line in several places into a musically satisfactory phrase. However, it is easy to flood the piece with too many.
- Several suspended notes are marked with 'S'. The program does not tie them to their parent note in the previous bar; this is left to the taste of the user.
- Some appoggiaturas may resolve upwards, some downwards.
- In LIPs 1 and 2 the program has elaborated the given notes into crotchets and quavers. The elaboration of each voice is independent of the other, but within the any one LIP each 2-unit cycle is given the same rhythmic motif and the same variations in pitch either side of the given pitch. This similarity accords with 18th century practice and emphasises the presence of the linear progression.
- Musically, the passing notes have smoothed the piece, the modest discord of the suspensions and appoggiaturas has added bite, and the treatment of each voice line independently has given the piece some limited contrapuntal interest.

Figure 71 is clearly incomplete as a piece of music. By adding passing notes to each voice independently of the others, there is the possibility of faults being created between any two voices, such as consecutive discords and parallel perfect fifths and octaves. However, I find that these occur in only a few places, and can be corrected by hand so I have not thought it worthwhile to write the additional code needed to detect and correct these automatically. The issue is more important when lots of auxiliary notes are added. In §14.2 therefore I explain the algorithm used to avoid parallels and manage discords in Program 10 when patterns of neighbouring notes replace single longer notes.

Figure 71: Framework of a short piece created from Figure 70 by adding passing notes, suspensions (S) and appoggiaturas (A) with Program 9. The three LIPs are highlighted in red.

14 Program 10: auxiliary notes & rhythmic decoration

Taking a 'helicopter view' of this suite of programs, Programs 1, 6, 5 , 7 and 8 in Group 1, used in that order, will create a type of hymn tune in 3 or 4 voices such as Figures 35 and 70, while Programs 9 to 12 in Group 2 will then place a rhythmic and melodic 'skin' over the bony chord skeleton of the hymn tune. Decoration is a rich and complicated matter because of the many forms it can take and the places it can be added. Figure 18 in §7 shows how Handel or his contemporary elaborated chords into a musically satisfying line in a violin sonata. The topic was taken up again in §10 and §12 where Figures 38, 51, 52, 53 and 61, 62, 63 show how intervals, scale passages and triads were elaborated by 18th century composers. Passing notes and suspensions created by Program 9 introduce some smoothing of voice lines, while as a by-product they add rhythmic subdivision of long notes. In considering how all these could be added to a harmonic skeleton, we need to have regard both to the pitches which decorate each single note and intervals between notes, and the rhythms assigned to groups of notes, always keeping these within a musical structure of phrases. As I have remarked previously, these programs should be regarded as a tool box in the hands of a musician.

In this section I first report some enquiries I have made into the variety of rhythmic patterns typical of 18th century pieces (§14.1). Later subsections consider the rhythmic division of long notes, both by repeating the same pitch and by adding upper and lower auxiliary notes, also called neighbouring notes. There follows a description of Program 10 with examples. This program was written in two stages. The first dealt only with embellishing a single voice moving in crotchets or minims, such as that created by Programs 2 ands 4. This was later generalised to deal with notes of other durations in up to four voices, with checks to ensure that gross discords and bad parallels are not introduced between voices. This later version uses the rhythmic patterns already created by Program 9 in the course of adding passing notes, and both versions invent additional rhythmic patterns as necessary, plus patterns of pitch variation, and assign them to one given voice line at a time to replace notes of a crotchet or longer by shorter notes.

14.1 Rhythmic decoration of a single voice line

I have explained that Program 10 was developed in two stages. I describe here the first version which will embellish a single voice line stated only in crotchets and minims by breaking up the given notes with a rhythm of shorter neighbouring notes. There are only a few ways in which this can be done without sounding notes which might imply the presence of another voice:

- by adding a rhythm on the one note,
- with a single escaped neighbour note (échappée),
- with a neighbouring upper or lower auxiliary note,
- with an ornament such as a mordent or turn,
- with one or more scale-wise passing notes on weak beats to the next given note (done with Program 9, §13).

If we were to sound a note a musical 3rd away from the given note, it could imply a triad. That is why any accented decoration of a single note can only move through a 2nd. The above options are illustrated in Figure 72. With pitch variations limited to 1 to 2 semitones at a time, complicated rhythmic patterns will in most cases not be appropriate. However musical interest can be imparted by replacing a simple scale in long notes with a repeated pattern of short notes, as in Figure 73 D.

As an illustration of the direction in which this is developing, Figure 74 is an elaboration of the opening of the hymn tune of Figure 70 up to the second cadence. These few bars do have musical interest, sounding rather like the opening of a baroque violin sonata or concerto grosso when played at metronome rate dotted crotchet = 80. I have *by hand* added some further rhythmic decoration, using repeated and neighbouring notes, copying some of the rhythms already generated by Program 9 in the course of adding passing notes. A few embellishments go beyond adding auxiliary notes: for instance, the imitation of the first bar in the soprano in the second bar of the tenor, and the roulade flourish in the top (violin) part at the second cadence. By breaking up the long notes with rhythmic patterns we add interest and forward motion to the music, while by having only a restricted palette of rhythmic motifs, the piece keeps its unity and the listener is led along. Notice that generally I have arranged for movement in only one voice at a time, to keep the texture simple and fairly obvious.

The time signature, chord type at each beat, the positions of cadences and the prevailing key at each chord – which will already have been decided at Program 6 and carried forwards into Programs 5 to 9 – are all input data to Program 10.

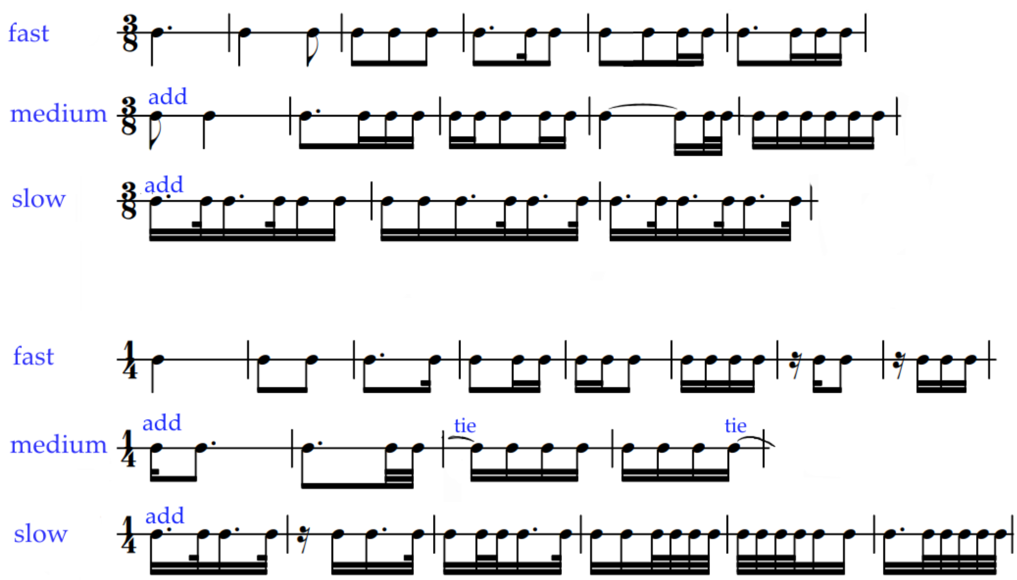

Figure 72: Elementary sub-motifs used by Program 10 in building rhythmic patterns. Upper panel of three: triple or compound time. Lower panel: duple or quadruple time.

Figure 73: A single voice A decorated by passing and auxiliary notes at B, and by escaped notes and rhythm at C and D.

When embellishing a single voice line, Program 10 follows this method:

1. store the parent note, the triads, the cadence positions and the local key (after any modulation) as created by previous programs in the suite,
2. note passages of 3 or more scale-wise steps in the same direction and of the same duration, over which a sequence could be applied.
3. use the rhythmic patterns created by the passing notes in Program 9 and, as appropriate to the required style, create a number of additional rhythmic patterns from the given sub-motifs.
4. the program has the facility to adjust the pattern length to suit several durations of given notes: crotchet, minim, dotted minim and semibreve, and their compound meter equivalents.
5. assign rhythmic patterns and pitch variations of ± 1 scale step. The user can set options to embellish or not the closing notes of cadences. Suspension and appoggiaturas are never embellished with auxiliary notes as that would be contrary to their purpose.
6. note the prevailing key at each chord and hence introduce accidentals as required to make the decorating notes consistent with the parent triad type and local key. Adjust to the ascending or descending forms of the melodic minor scale where necessary.

Figure 74: Illustration of adding suspensions, passing and auxiliary notes (the latter by hand) to the first 9 bars of the hymn tune at Figure 70. Metre changed to 12/8.

The user can choose to save the created rhythmic patterns to file for use by the same or another program. The program can read a saved or user-specified set of patterns from data statements. This allows programs in the suite to use the same patterns in different voice lines, in compound melody and/or in inserted sequences, thereby giving some rhythmic unity to any longer piece made by stitching together several sections.

To provide a simple example I have used the first version of Program 10 to decorate the steady minims in row C of Figure 9, §5.2 which was produced by Program 2. Figure 75a copies this line though in the treble clef. Cadences are here defined simply as places with the adjacent notes D-C or B-C, as marked with red spots. The second note of each cadence is retained as a long note. The green lines show scale-like movement which the program will elaborate with the same rhythmic and pitch pattern to form a sequence. The exception is where one note in the scale is a cadence note, in which case it is not decorated, as is the case at the first and second red spots. Figures 75b is one examples of the output from Program 10 in decorating this line of minims, transposed up one octave. It is at a fast dancing tempo in compound time where each bar lasts a dotted minim. The program has created only three decorative rhythm patterns, each one bar long, first stated in bars 1, 2 and 6. Pitch variations about the given note are only through the interval of a diatonic second.

Figures 76 is an alternative elaboration of the same voice line in simple time appropriate to an *adagio* or *grave* tempo. Here the program has provided eight different patterns. Because selection of rhythm and pitch motifs which seed the bar-long patterns is made by random numbers, no two runs of the program produce the same output, and here they are significantly different. To sound effective this piece probably needs a steady tread of a supporting bass and continuo, and careful attention to articulation and phrasing. With effective bowing, they might sound satisfactory on a

Figure 75: Decoration by division of a line of minims (a) into a fast compound time (b).

violin, perhaps with some ties, rests and passing notes added by the player. The violinist would also wish to change some notes to taste – for instance to limit the number of successive repetitions of the same pitch, or to enforce passing notes. This can be done within the program suite by a combined use of Program 9 and 10 as described below.

I will mention a few more details of this first part of Program 10. The given input notes are assumed to be in simple time (2-4, 3-4, etc.). The time signature and the total number of notes are input. The user can choose to have the output in the same simple time as the input, or to have it converted to compound time (e.g 2-4 → 6-8, 3-4 → 9-8) in which case the durations of the given notes are increased by 50%. The program is primed with about 18 rhythmic motifs each lasting a crotchet (simple time) or dotted crotchet (compound time). These are arranged internally in order of increasing complexity. The user chooses a nominal tempo at which the piece will be played, and the program proposes to create a certain number of 2-beat rhythmic patterns depending on the tempo, with Fast having only 3 patterns, and Slow perhaps 8 or more. The user can change this number, but once it is specified, the program will randomly select 1-beat motifs from the list available to that tempo and combine them pair-wise into the required number of 2-beat patterns.

The program also has lists of pitch deviations from a given note stored internally. For instance, if a given note is to be divided into 4 notes by some rhythmic pattern, the list might be $[0, 1, -1, 0]$ or $[0, 1, 0, 1]$. $[0, 1, -1, 0]$ means that the first and last notes of the 4 are at the same pitch as the given note (0 variation), the second is one note higher in the diatonic scale (+1), and the third one note lower (−1). A trill starting on the upper note could be created by $[1, 0, 1, 0, 1, 0, 1, 0]$ when 8 notes replace the given one. The program then steps through the given list of notes, assigning rhythmic patterns and pitch variations at random from the selected set. If it encounters a scale passage in notes of the same duration, it will assign them the same rhythmic and pitch variations to create a musical sequence – a genuine sequence in Forte's restricted sense of the word. In simple time a dotted minim will be replaced by a 2-beat rhythmic pattern then a 1-beat motif, so every dotted minim will have a different rhythm. I judge that this is not unmusical. The output voice line

Figure 76: Alternative elaboration of the line of minims at Figure 70a for performance at slow tempo.

is saved to file, and the user is offered the option to apply the same set of rhythmic patterns (but not the same patterns of pitch variation) to either the same or another voice line.

It will be clear that the 'music' in Figure 76 leaves much to be desired. This is because of there are too many rapidly repeated notes and no scale passages longer than a 3rd. In many places the adjacent notes do not move smoothly from one to the next – we feel the need to change some to be passing notes. Perhaps the output of this first versions of Program 10, suitable edited, might form the middle voice of a piece by usefully adding rhythmic interest. We might say that Program 10 decorates a hymn tune by emphasis on the rhythm, with pitch variation as a by-product, whilst Program 9 treats rhythm and pitch variation with the opposite emphasis. Clearly, however, real music requires the two to be tastefully interwoven and that is addressed to some extent by the later, second version of Program 10..

14.2 Management of discords

The main change to the first version of Program 10 – needed to have it deal with four voice lines – is to build in checks which prevent parallel fifths, parallel octaves. Avoiding these poor, unmusical note combination is something 18th century composers must have done almost instinctively, but it poses a challenge when the processes are to be converted into computer code.

In the 18th century and early 19th century classical styles, concordant intervals were much preferred to discordant ones. In order to put some numbers on this, I have examined all the vertical intervals in one two-part piece by Bach, the second section of the *corrente* in G from keyboard Partita

5. An extract is given in Figure 77, showing the mixture of scale and arpeggio passages typical of this lively piece. From bars 32 to 62 I noted the intervals between the upper and lower voices as 2nd, 3rd, 4th, 5th, 6th or 7th, and recorded the few augmented 4ths (denoted 4^+) and diminished 5ths (5^o) separately. For instance, the intervals in the first bar are 3, 0, 6, 3, 6, 2 where 0, equivalent to 8, denotes unison or an octave. There are 183 such intervals each lasting one semiquaver. Although this is only one piece, it is probably typical of Bach's choices. Figure 78 plots the relative frequencies of the various intervals. (Compare this with Figure 8 which shows the frequency of horizontal intervals in one voice lie.) 3rds and 6th make 25% and 27% of all vertical intervals respectively. 4ths and unison/octaves are equal at 11%, 5ths and 2nds are at 8%, 4^+ combined with 5^o are at 5%, and 7ths at only 3%. However, since a 3rd and its inversion, a 6th, sound much the same, it is fair to combine them and observe that they make up 51% of the intervals. Similarly 2nds+7ths make 11%. This may in part be a result of Bach's generous use of dominant 7th chords. However, 5ths cannot be combined with 4ths since the 5th was considered a concord, but the 4th a discord is two-part writing.

We might expect the discords to sound most of the time off the beat – that is, on the 2nd, 4th and 6th semiquaver of each bar. There is some evidence for this in this piece. If 3rds, 6ths, 5ths and octaves are grouped as 'concords' and 2nds+7ths as 'discords', we obtain the frequencies listed in Table 14. I have listed the 4ths separately even though they are a type of discord. Concords are only 5% more likely to occur on the beat than off, not a large difference. More significantly, in almost

Figure 77: Extract from the *corrente* in Bach's Partita No. 5 in G major, BWV 829, (Bärrenreiter edition). The red areas mark consecutive discords.

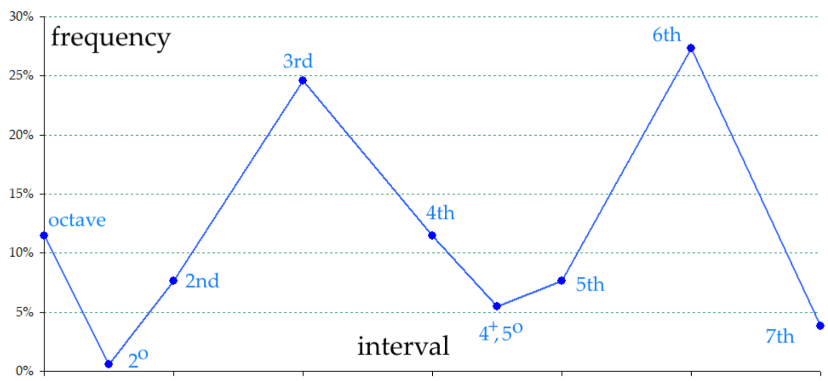

Figure 78: Frequency of vertical intervals in the second section of Bach's *corrente*, Partita in G.

on beat concord	on beat discord	on beat 4, 4^+, 5^o	off beat concord	off beat discord	off beat 4, 4^+, 5^o
38%	5%	7%	33%	7%	10%

Table 14: Relative frequencies of concords, discords and 4ths sounding on and off the beat.

every bar, within every quaver beat there is at least one concord. There are only two instances of consecutive discords; I have highlighted them in red in Figure 77. They occur between E and F♯, B, C and C♯ in scale passages moving in contrary motion – another example of our ready recognition of step-wise motion and willingness to let it override minor aural conflicts. A crucial point is that *there is no example of any consecutive discords in parallel or similar motion.*

14.3 Example of a short piece

Returning to the development of Program 10 into its second version, its code actually follows directly after Program 9 within the same piece of software. Though 9 and 10 were originally conceived as two separate, stand-alone programs, it has proved sensible to insert the passing notes into the hymn tune first, then provide the resulting voices lines directly as input to Program 10 without having to save them to file and then reload. The individual rhythmic patterns attendant to creating the passing notes are automatically identified and passed to Program 10 within the same piece of computer code. More rapid rhythms are added according to a slow tempo to constitute a set of candidate rhythms.

As an example of a whole short piece, I have decorated the example Figure 71 (itself created from the hymn tune in Figure 70) with auxiliary neighbouring notes using Program 10. The result is in Figure 79. The user operating the program sets the option flags which determine whether suspensions, appoggiaturas, dotted rhythms, etc. are to be allowed. When auxiliary notes are being added, the user can select which voice to embellish, or else have the voice selected at random, but in any voice the note is selected at random from all eligible notes. The only ineligible notes are suspensions and appoggiaturas, notes shorter than one crotchet (dotted crotchet in compound time) and, optionally, the closing note of a cadence.

The program compiles from a library of rhythm patterns a short list which could be used. Having randomly selected a note, the program randomly selects a rhythm from this short list and also a pattern of pitch variations having the same number of notes. This combination is tentatively applied to the given note and checked against those notes which overlap it in each other voice in turn. The check involves computing all the sequential musical intervals between the two voice over the short time interval when they would sound together. A count is made of parallel octaves, parallel fifths (irrespective of their being perfect or diminished) and of consecutive discords. The rhythm+pitch pattern between these two voices is acceptable only if there are no such parallels or consecutive discords. For this purpose intervals of a 2nd, 7th and 4th are counted as discords. The pattern is acceptable overall only when it is acceptable in all three voice pairs (in four voice writing), in which case it is copied from the holding array to the output array, replacing the original note. If not acceptable, a random choice of made of another rhythm+pitch pattern and the checks repeated. This cycle is allowed to continue for six selected patterns, but after that the program abandons the attempt and randomly selects another candidate note in the same voice. I found in early tests that it was better not to embellish any given note by dividing it into just two notes, since if they were at different pitches the ear would be confused as to which was the harmony note. I also decided that dividing notes but keeping them all at the same pitch, while adding some rhythmic interest, generally just became tedious and added no musical value.

Figure 79: Addition of auxiliary notes to produce an organ prelude from the hymn tune with inserted passing notes and suspensions of Figure 71.

The longer notes are embellished with auxiliaries one at a time, and the user can stop after each note to save the output to file. This gives the user control over the number of embellished notes, though not their position or pattern. I found it best if the elaboration were restricted to one voice or perhaps two, except for short durations where three voices might be elaborated to enhance tension. There are few auxiliary notes in the bass as these can sound muddy and indistinct. I have allowed the cycles of the LIPs to be elaborated individually, that is, without requiring that they have exactly the same rhythm and pitch patterns. To the ear each intervallic pattern is still quite clear, while the surface differences between cycles add to interest.

I have found it good to decorate, say, six notes in a run then save the output, convert it via

Program 3 to Lilypond and hence with Frescobaldi to pdf and MIDI files, and assess by ear whether it seems musically interesting. From half a dozen such saves you can pick the one which sounds best. Too few auxiliary notes and the sound is bare; too many and it is a confusing jumble. It is desirable to keep the cadences clear, so the corresponding flag was set to ensure this. Figure 79 is the output I chose. Having got this largely computer-generated output, I made a few by-hand changes, mainly dotting some quaver pairs and adding a few rests to let in some air. I also transposed it to d minor so that all bass notes could be played by the pedals of a real organ. I felt the piece was starting to sound like an antique improvised organ prelude from Pachelbel's time, though with a few syncopated rhythms which I have left as the computer created them. It does not have the recognisable themes of Haydn or Mozart. I then transferred the MIDI file with these by-hand changes to the free GrandOrgue software[15] and recorded it as an mp3 file on my web site at *www.mathstudio.co.uk*.

In another run with Program 9 I set the flags to emphasise the suspensions and appoggiaturas. This piece is also suited to a slow, stately pace and the falling resolution of the suspended notes and appoggiaturas from the first to the second beats of the bar gives the music a melancholy, sighing character suited to a funeral march. Rachmaninov's music is full of notes falling by step and creating a similar melancholic, yearning sound world. I do not consider any of these variants of Figure 70 to be worthwhile pieces of music, but nor are they dreadful. They retain the overall arching shape imposed by the guide curves of Program 5. The thickened texture towards their middle bars increases tension, after which some sparse chords with crotchet rests in the last LIP can bring the piece to a sensible conclusion. These pieces were short enough not to outstay their welcome.

15 Piano accompaniments and simple musical variations

In this section we consider the decisions a musician might make in elaborating a few chords to form a variation, as might have been done in the sets of variation on a theme in the 18th century. Related to this are the broken chord accompaniments used in piano music from the mid-18th century up to Rachmaninov's time. This leads to Program 11 for implementing broken chords above or below a given single vocal line.

To set the scene, let us examine a few examples which I have invented to elaborate a 4-chord phrase in C major with the harmonic sequence Ib IV V Ia. The examples are musically trite, but determining the algorithms behind them is not. Figure 80 states the 'theme' and lists 7 distinct variations, two of which appear in slightly different guise. Note these points:

- Variation 1: an escaped upper auxiliary note is suspended into bar 2. Another suspension from bar 3 to bar 4. Singable and playable by many instrumental groups.
- Variations 2a, 2b: Elementary compound melody by alternating the notes in S and A. Passing notes added in 2b. 4 voices reduced to 2 or 3.
- Variation 3: 2 parts only. A melody has been created in the treble by adding passing notes to the compound melodies of Variations 2. Octave transfers of notes D and E in bars 3, 4 restore smoothness to the line and hence improve the melody. Playable on keyboard any by many instrumental duos.
- Variation 4: Outer voices unchanged, but A and T combined by passing notes and a few auxiliary notes. Repeated rhythmic pattern gives unity.

[15]GrandOrgue is a virtual church pipe organ available at *(sourceforge.net/projects/ourorgan/*. Modartt's Organteq is more modern software which simulates the organ sound from mathematical models.

Figure 80: Seven variations (not computer generated) on a 4-chord progression.

- Variation 5: 2 parts only. Compound time with steady triplets in the bass. A melody has evolved in the treble, with the dotted notes adding rhythmic interest. Suitable for keyboard or instrumental duet.
- Variations 6a, 6b: Pedal C and B in the upper part. 6a and 6b differ only in the placing of the pedal note on or off the beat. Bass part, now a melody, needs to be louder to sound above the upper. Melodic line in bass has been carefully shaped, and semiquaver pair adds interest. Suitable for keyboard or violin + cello duo.
- Variation 7: The Alberti bass gives a keyboard style which Mozart would have recognised. I have used Mozart's familiar rhythm in the right hand.

Behind these broad points are many subsidiary choices which the musician could make. Some guiding principles are:

1. to maintain mostly concordant intervals, though allow a few discords, especially anticipated discords, to drive the music forwards,
2. to shape a melody, making choices to prevent any meandering around 2 or 3 adjacent notes, or jagged leaps,
3. to maintain sufficient rhythmic similarity from bar to bar that the music sounds of a piece,
4. to allow some rhythmic interest and emphasis with dotted and short notes,
5. to bear in mind how the music will be realised, whether by voices, keyboard or instrumental ensemble, so that it is idiomatic.

In terms of algorithms, it seems not unreasonable to identify four types of variation technique and write a separate computer code for each:

1. elaboration in 4 voice writing using only auxiliary notes and suspensions,
2. compound melody by alternating notes between 2 or more voices, with consequential reduction in the number of parts,
3. pedal notes alternating with others,
4. specific keyboard devices including Alberti bass and wide arpeggios.

The first of these is addressed by Program 10, described above. The second and third are discussed in §16. Here I describe Program 11 which deals with the fourth case: arpeggios or more general broken chord patterns on triads, combined with a single vocal line. The input to this algorithm is a sequence of chords described only by their Roman numerals I, IV, V *etc.*, and a given bass or treble voice line which may contain passing or auxiliary notes. This vocal lines could have been generated by Program 9, or 9 and 10 together.

The concept of Program 11 is illustrated in Figure 81. We suppose first that a bass voice is given, here shown as semibreves The triads from which this line was derived are also given, here shown as triads of C major in crotchets extending above the bass. If the bass voice actually is a note of the current triad (C, E or G in this example), the super-structure of triad notes starts from the bass. If the bass is a non-triad note, it starts at the lowest triad note above the bass. This collection of given bass and triad superstructure can be broken up rhythmically in many ways, from simple arpeggios up and down the extended chord to complex jumping from note to note which may in

Figure 81: Progression of seven bass notes (semibreves) with the triad chord of C major above. Last two bars show the C triad below a given treble in semibreves.

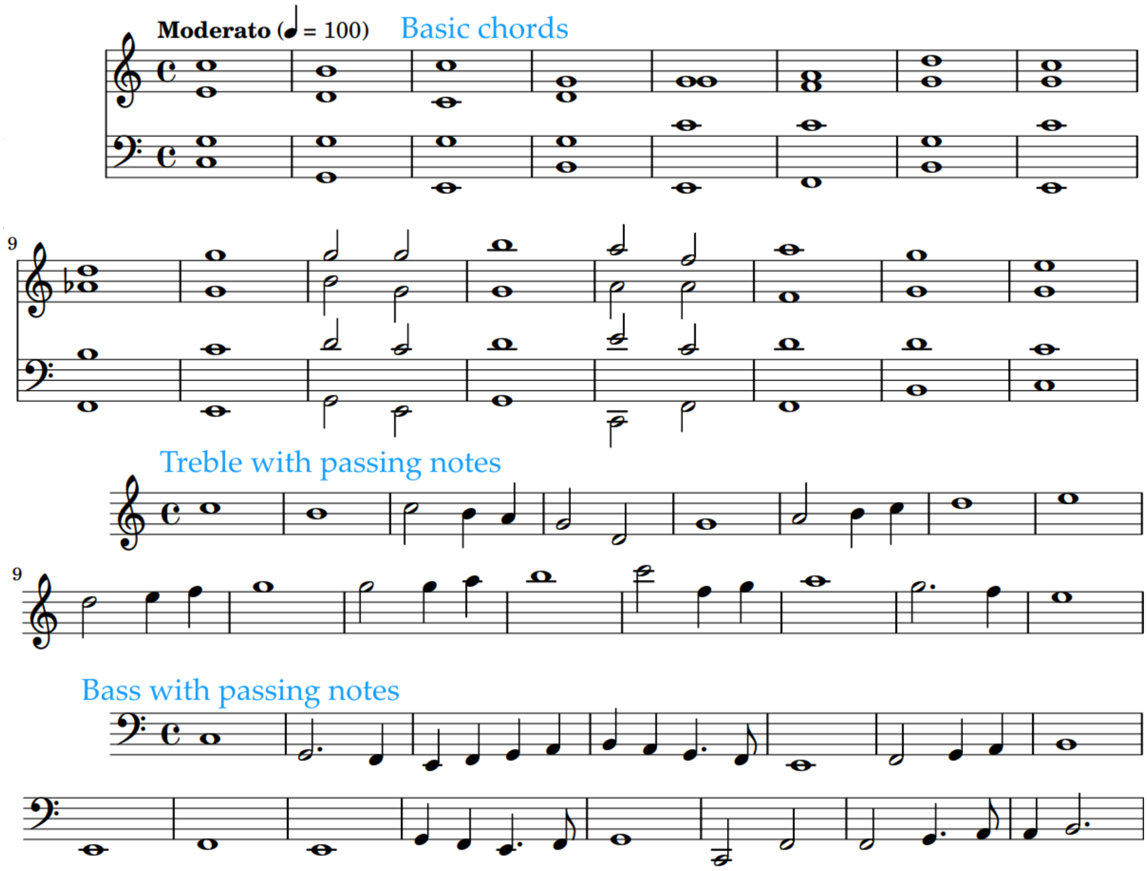

Figure 82: Source materials used in the illustrations in Figures 83, 84 of how Program 11 can elaborate a given voice line and a triad progression.

some cases sound like a simple tune. The last two bars of Figure 81 show that the scheme can be inverted so that the chord is below the given treble voice.

To have some simple starting material on which to develop then illustrate the program, I used the earlier programs in the suite to create the trivial sequence of 18 chords in Figure 82 A, and decorated the treble and base lines with passing notes using Program 9, Figure 82 B, C. In Figures 83 and 84 I have collected several examples of the variations which Program 11 can generate from one or other of these sources.

Figure 83 has four examples of decoration of the treble voice; only the first few bars are printed. In A and B the given treble voice forms the top notes, while in C and D it forms the bottom ones. Note the oddly mixed rhythms and the two-note chords in Figure 83 D. The given treble note

Figure 83: Examples of treble or right hand keyboard parts generated by Program 11 from the voice line at Figure 82 a.

falls consistently on the beat only in D. Each of these could be played effectively on an 18th century fortepiano while . A and C would probably work on the violin, flute or clarinet.

Figure 84 lists the first few bars of nine variant bass lines derived from the source bass with passing notes in Figure 82 C. In all cases the given bass note is the lowest in each group. A and B are classic Alberti basses. A is the simplest rocking between notes of the bass and triad, while B has some upper auxiliary notes. C is a rhythmically elaborate left hand part with the bass note sounding last in each group. This would probably not be found in music until the 19th century. D is a simple triplet accompaniment, while E adds 2-note chords. F, G and H have chords mixed with single notes in different rhythms. Finally I could be the furious left hand part to a piano study from Chopin's time.

15.1 The algorithms in Program 11

The basis of the algorithm is Figure 81. Here are some comments on the options built into the program which give limited control over the output.

As noted above, the input data are i) the sequence of triads in Table 9 notation and ii) the bass or treble line as MIDI pitches; this in general will contain passing and/or auxiliary notes. No note should be shorter than a quaver. The sequence of triads can be transposed by also inputting an array which defines the prevailing key at each triad (see §8.1 for how the programs give effect to modulation). The program is told the unit of duration – crotchet or minim – and it then divides that unit into shorted notes by a choice of ways. A pattern of pitches is then assigned to the rhythm, and the rhythm+pitch pairing used as a unit to paste in the place of a crotchet or minim.

A number of control flags are set which determine:

1. whether the broken chords are to be above or below the given voice line,
2. how many different crotchet-length and minim-length rhythm+pitch units are required,
3. whether the unit of duration which is rhythmically divided as always a crotchet, or else a minim if the note is long enough,

Figure 84: Beginning of nine bass accompaniments generated by Program 11, decorating the bass line in Figure 82.

4. whether triplets are forbidden, or whether they must be used, or can be randomly assigned,
5. whether an Alberti base is required,
6. whether 2-notes chords are allowed,
7. whether the same rhythm is to be kept while the pattern of pitches assigned to the rhythm changes.

Point 3 affects how notes lasting for two or more crotchets are treated. With the flag set to 'all crotchets' all notes of a crotchet or longer are broken into crotchet lengths and a rhythm+pitch unit of crotchet length pasted over each. With the flag set otherwise, minim-length units are used where possible. You can see in the bass parts in Figure 79 that at A, B, F, G, H and I minim-length units have been applied, but at C, D and E they are only crotchet length. 2 distinct crotchet patterns are present in C. Some flags constrain the number of notes into which a crotchet or minim may be divided. For instance, with triplets a crotchet can be divided only by 3 or 6, and with an Alberti bass only into semiquavers per crotchet (though they could be dotted). Once the flags are set by the user and these constraints applied, the program selects the rhythm patterns at random and assigns pitch sequences at random from an internal database of rhythms and pitch patterns. The output is a single musical line to file. As with all these porgrams, Program 3 must then be run to convert the coded notes, listed by MIDI pitch and duration, into a Lilypond file and hence engraved using Frescobaldi.

To close this section, note that it is possible to take the output from Program 11 and feed it into Program 9 for more passing notes to be added. An example is Figure 85 where some broken chords have been changed to scales by filling in every interval of a third. (One of the conversion programs in the Program 3 set is needed to convert the output from Program 11 to DATA statements suitable for input to Program 9.) Alternatively, if applied to a treble voice, this will smooth the melodic line.

Figure 85: The bass in Figure 79 D, generated with Program 11, further elaborated using Program 9 by inserting passing notes into every interval of a 3rd. Original in upper line.

16 Elaborations which combine two or more voices

So far all decoration has been limited to one voice at a time, as if its notes were independent of those in the other voices. This section deals with a program which attempts some of the more subtle techniques of melodic decoration within a given harmonic framework, all of which involve notes from two or more voices being sounded in succession by only one voice. In creating a melody by elaborating a succession of chords, any melodic intervals other than an octave and ±1 diatonic steps require the splicing of two or more voices. We therefore pick up practices most of which were illustrated in Figures 51 to 55, namely

- arpeggiation – the spreading of a chord into three of more notes played in succession,
- octave leaps in scale passages, whereby one or more notes are transposed by an octave up or down,
- alternation between a moving voice and a fixed voice. The fixed voice is often called a ‘pedal’ by analogy with a bass note being continued for a bar or more on a organ by holding a foot down on the pedal board.
- compound melody in which two or more voice lines sound as if they have been welded into one by alternation of notes between the two voices.
- expanding a triad into a musical phrase.

These are not generally independent types of decoration, since in some passages it is reasonable to ascribe the composer's choice of notes to more than one of these. Bear in mind that the model I use is that the starting material is a hymn-tune-like progression of chords in 3 or 4 voices, possibly with linear intervallic patterns inserted, created by Group 1 programs. In any one chord we only have its 3 or 4 notes at their given pitches to work with, though the triad type of each chord is also known. Decoration must respect both the instantaneous harmony and the melodic voice leading. Any interval of 3rd or greater will imply a triad, and this triad must be consistent with the background triad type: I, IV, V, vi, *etc.* It is to avoid implying triads that Program 10, §14.2, applies pitch variations of only ± 1 diatonic step from each given note. In this section we introduce wider intervals by linking one voice with another.

In order to develop algorithms for the above types of musical decoration, I need a deeper analysis of 18th century practice. Musical textures created with the above devices occur mainly in instrumental music, the composer having regard to the particular instrument on which the piece will be played. The notes need to fall under the player's fingers. Generally scale passages can be played on all instruments. However, rapid alternation of notes across intervals as wide as one octave are more readily played on a keyboard or violin-family instrument than on woodwind. Of course, even string instruments can play chords after a fashion, the bow rapidly crossing the strings and perhaps holding the upper two notes together. Here the arpeggiated chord must be written so that a separate string can play each note. I will have keyboard and violin-family instruments in mind. The types of decoration used and the percentage of time that each occupies in the piece will depend on the intended style and type of piece, as well as on the instrument. For instance, we find more stepwise movement melody in allemandes and more arpeggiation in gigues.

16.1 Arpeggiation

The simplest and perhaps most obvious way to splice several voices into one is just to play one note of each in sequence, as an arpeggio. To show the potential problems I demonstrate in Figure 86 a an obvious mechanical but unmusical arpeggiation of the hymn tune at Figure 35. Since only notes of the original chords are used, the voice leading remains satisfactory. Duration of notes has been adjusted so that no notes are missed out. Thus each crotchet is divided into four semiquavers and each minim into four quavers. With the dotted minims two notes of the chord have been repeated. The results could be played on a keyboard, but would probably sound austere precisely because the musical line is disjointed, with no smooth stepwise, singable intervals. It might sound better on a 'cello or viola if the chords were rewritten to fit into the instrument's compass. The version at Figure 86 b is playable on a keyboard or by two one-note instruments. It retains the bass unchanged and has arpeggiated the upper three voices, while introducing some rhythmic patterns to accommodate the different note durations. It is more musically satisfactory than the upper panel.

In Bach's suites for solo violin and solo 'cello we find several movements which are built largely from arpeggiated chords. One example is the Corrente in Figure 87. It must be admitted that this style of music is not to everyone's taste. Similar movements by Bach include:

1. Doubles 1 and 2, Partita in b minor, BWV 1002,
2. the Allegro, a minor solo violin sonata, BWV 1003,
3. Prelude and Courante, 'cello suite No. 3 in C,
4. Prelude, 'cello suite No. 4 in E♭ – almost entirely arpeggios,

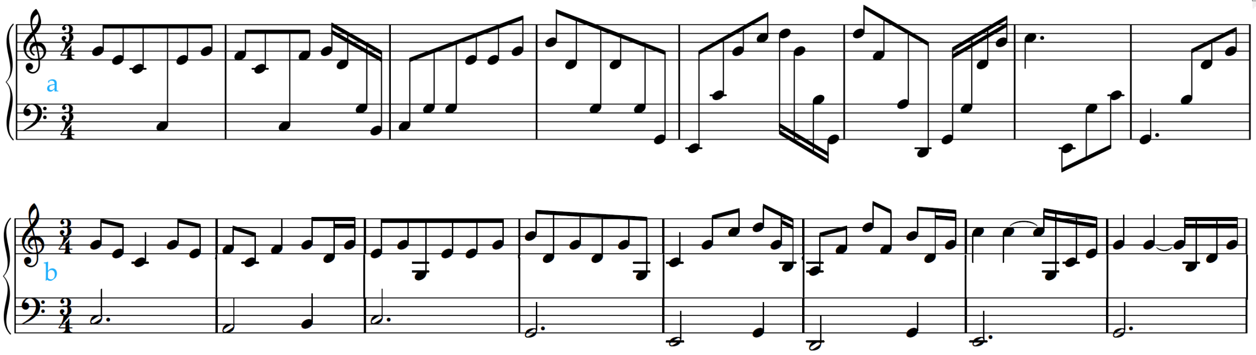

Figure 86: a) a naive way to convert the chords of Figure 36 to a single voice by arpeggiation. b) musically improved version which arpeggiates only the upper three voices and adds rhythmic variety.

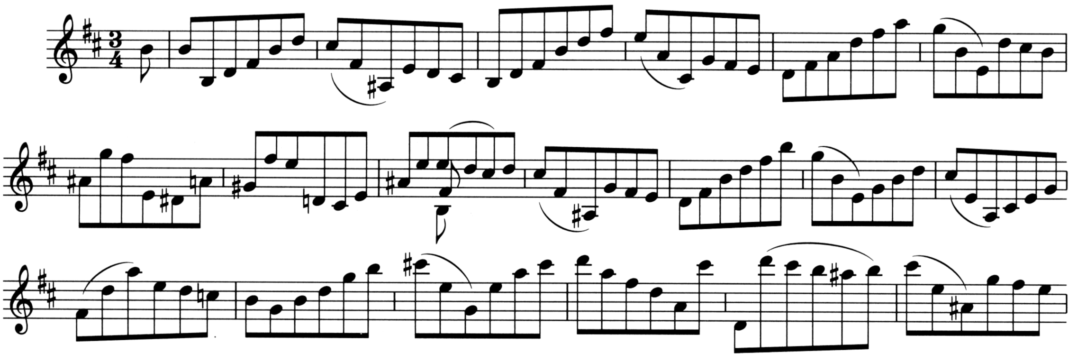

Figure 87: Opening bars of the Corrente from Bach's Partita 1 in b minor for solo violin, BWV 1002. (Wiener Urtext Edition), showing frequent and wide arpeggiation.

5. Sarabande, 'cello suite No. 5 in c,
6. keyboard Preludes in C and G, WTC I.
7. Gigue from the French Suite No. 5 in G.

With few exceptions (one being the Prelude to 'cello suite No. 4), the arpeggios are monotonic up or down – they rarely jump up-down-up-up-down-*etc.* in a zig-zag or seemingly random manner. In works for solo violin the range from lowest to highest note can be up to two octaves, though one octave+5th is more often the maximum. The harmonic rhythm is quite slow in arpeggiated pieces so we can find one chord, in quavers or semiquavers, lasting one or two bars and incorporating 6, 8, 12 notes or more. Once a pattern of arpeggiation has been started, Bach will continue it for many bars, as in the Prelude to 'cello suite No. 1. In many pieces voice leading will not be followed rigorously from chord to chord, though the relation of prominent notes to one another over several bars can usually be discerned – a major concern of Schenkerian analysis. Scarlatti also uses arpeggios over two or more octaves in his keyboard sonatas.

Except in a few instrumental preludes referred to above, few of Bach's works have long periods made wholly of arpeggios. If the arpeggios are filled in with passing notes, the composer produces a scale-like passage so it is usual to have some mix of arpeggios and scales, if only to give the ear and mind a break. The cello allemande, Figure 55 A, is an example of a piece where arpeggiation is restricted to a few places within generally scale-like passages, though the arpeggios are given emphasis by starting on the first beat of the bar. The relative amount of arpeggiation to stepwise movement is clearly a characteristic of the type of piece. By transposing some notes up or down by an octave, some of the disjoint nature of the line can be relieved, and by inserting passing notes the ratio of arpeggiation to stepwise motion can be adjusted. One issue to deal with is how to treat chords of unequal duration, assuming the harmonic rhythm is not to be abandoned by making each chord last a minim. Clearly some notes in some chord may need to be omitted. Dealing with this complexity is discussed further in the section on compound melody, §16.5 below. Indeed, the upper staff of Figure 86 b comes fairly close to compound melody.

16.2 Melodic extension of a single triad

This feature of musical elaboration is the process of extending a single chord into a whole musical phrase. Broadly, the notes of the chord form a framework positioned at the stressed beats of the bar and connected by some passing notes and padded out with some transitory related chords. By such methods one chord can be made to fill one, two, three or more bars with a musical phrase. The phrase may start and end on the parent triad, or may move from I to V, or similar strongly directed progression. This is the stock-in-trade of 18th century composition. So very many overtures, sonatas, symphonies, concertos open with a theme which is a rhythmical arpeggiation of the tonic chord. This was done, of course, to establish the tonic as the sound of stability, the aural point of departure and return, so characteristic of and essential to western art music of that time. The topic of extending a triad is closely related to §11.1 on what makes a good tunes, and to §16.1 on arpeggiation, and indeed

Figure 88: Five examples of a triad extended to form a phrase lasting two or more bars.

Program 11 will accomplish this, though without finesse. Some recapitulation of the principles here is hard to avoid because the aspects of music are only separable for the sake of discussion, not in the listening.

As always it is best to quote some examples. Look back at Figure 60, the opening of a keyboard sonata in C by Haydn. My annotation marks a progression of triads I ii V I, yet the treble line has few stressed notes which are not C, E or G. The whole phrase is a sophisticated expansion of the C major triad. Figure 88 a is from another Haydn piano sonata, from 1778, where D major lasts for 8 crotchet beats before adding the 7th to fall to G major. At the opening of Schubert's a minor sonata D784 the only stressed note not from the a minor triad is D♯. At Figure 88 c , from the scherzo in sonata D575, Schubert has spread a G major triad over 15 beats. D major may be implied at the start of bar 3, but the note D is in the arpeggiation of G major. The two examples at d by César Franck both occur in a short carol-like piece from *L'Organiste*[16] and respectively show the expansion of E♭ and g minor triads.

A further memorable example of a few basic triads being elaborated into several bars is the opening theme of the violin sonata in e minor, K304, by Mozart, Figure 89. This beautifully crafted line for the violin starts at E above middle C, rises rapidly then falls in a cascade back to the starting note. After that it rises up the scale to B using a different rhythmic figure. The whole 12 bars are constructed only from the triads on e minor and B major with added 7th, as the analysis in the lower stave shows. Note how from bar 9 the bass note B is held in memory until the end of the phrase, as the tied chords indicate. This is a feature of all of the examples of compound melody I have cited; that the fragments of voice lines stay with the listener and need eventually to resolve correctly.

Figure 89: Opening of Mozart's violin sonata in e minor, showing (upper stave) arpeggiation and compound melody created from only two triads, e and B7.

The implications of all the above examples for a computer algorithm are that it must be flexible enough to allow for the many possibilities of arpeggiation, octave transfer, pedal notes and compound elaboration of the given triads. The starting point is a hymn tune which Programs 1, 6, 5 and 7 together have created. These essentially define only a harmonic rhythm, and have been created with the restriction of using only 4-bar blocks. So far only Program 10 breaks up this 4-bar structure by inserting linear intervallic patterns. Program 12 needs to identify individual notes, chords and pair of chords within the given hymn tune which are suitable for elaboration, list the options available at

[16] Piece No 12 in Easier Piano Pieces No.29, publ. ABRSM.

that position, and select amongst them. This may be done using random numbers to select options at selected triads, with the user having the option to override these and so steer the development of the music. The program needs to allow extension of selected single chords which initially occupy only one bar or one minim into structures lasting two or more bars. This can be done through arpeggiation of the one triad or by inserting another triad as, for example, I → I - V - I or I → I - IV - I. Where a new melodic line is created from the given chords, due care must be given to controlling its shape, perhaps again using guide curves or 'rules' of horizontal progression as in §5.5. Clearly there is wide scope to change the time signature and melodic rhythm. However, this freedom to deviate from the initial given hymn tune must be held within bounds, because the hymn tune has itself already been given a shape by the guide curves applied in Program 5. Therefore it may be necessary, or at least helpful, to return to the same note in elaborating a chord, as Mozart has done over the first eight bars of Figure 89. Program 9 will be used to insert passing notes. Furthermore, the music must be written with particular instruments in mind. With these challenging concepts to guide I have written the software.

16.3 Octave leaps and transfers

Figure 52, §12.1 gives examples of the composer breaking a scale by displacing one or more notes by an octave. This creates an interval of an octave if the note is repeated, and a 7th if the jump occurs within a scale. Figure 90 gives more examples from Bach's keyboard and string writing. What is the purpose of these? They may be

- to fit the scale passage within the compass of the instrument,
- to restore the voice to the pitch position which it had previously occupied but departed from significantly,
- to move a voice in pitch so that another voice can enter,
- to give emphasis to some notes to increase the musical interest,
- to give the illusion of a second voice,
- to sound a deep note in the bass,
- to convert an augmented interval into a diminished one, e.g. A♭ → B becomes B → A♭..

The note following the leap invariably lies inside the interval defined by the leap. I distinguish a leap through an octave from one through any other interval because transposing through an octave cannot suggest a harmonic change. However, we could also regard a leap of 6th as equal to (ne octave − 3rd), so pointing out that the voice has not moved to the closest note but to one with the same pitch value an octave distant.

The examples show that the jump through an 8ve, 7th or 6th can occur at any position in the bar and on any of the four notes in a quaver or semiquaver group. However, a leap by octave (8ve) occurs more frequently from weak to stronger, with the second (usually upper) note landing on a beat. In the small sample in Figures 52 and 90 there are 4 upward and no downward 8ves, 9 upward and 4 downward 7ths, and 2 upward, 2 downward 6ths. These values imply that downward movement is more usually by a chain of small intervals. In terms of a computer algorithm, it would seem appropriate to define upper and lower pitch bounds for each voice and to make an octave jump if the voice (perhaps transformed to a scale-like passage by passing notes) attempts to go outside these.

Figure 90: Four samples from Bach showing octave jumps and transpositions. a, b, c from French Suites: a) Suite 1, Corrente, b) Suite 2, Minuet, c) Suite 4, Gavotte, d) Cello Suite 1, Minuet 1.

16.4 Pedal notes

The original concept of a pedal note, or pedal point, was as a long, low bass note played on an organ. It begins as the root of a Ia or Va chord, and is held by the foot while the chords above change, creating a series of discords, but eventually returning to the Ia or Va chord when it again forms the bass of the harmony. The musical effect can be powerful in creating tension and expectation through its discords, especially when the note is indeed sustained by a instrument like an organ, as in the closing bars of BWV 547, Figure 91. This is a prime example of the original pedal concept. In Bach's WTC Book I there are sustained I and V bass pedals in both the prelude and fugue in C, at the beginning of the preludes in E and f, and in the closing bars of the fugues in c and c♯, and the preludes in E♭, G, g and a.

Figure 91: A long tonic pedal ending the organ fugue in C, BWV 547.

Figure 92: Three examples of bass pedals in Bach's writing for a) keyboard, b) solo violin, c) solo cello. From a) Prelude, English Suite No.2 in a minor, BWV 807, b) fugue, Sonata for solo violin in C, BWV 1005, c) Prelude, cello suite No.3 in C, BWV 1009.

If the instrument cannot sustain the note as a continuous tone, it can be held by repeating it, by a long trill (e.g. WTC I prelude in g) or by repeated alternation with moving notes in other voices. A repeated note in the bass, acting as pedal for a few bars, occurs at the beginning of the WTC I preludes in d, G and a. In the well known prelude in C from WTC I, a bass pedal G starts in bar 24 and is held for 8 bars, to be followed by a bass pedal C which last for 4 bars to the end of the piece. It sounds on the first and third beats of the bar, where the stress is greatest. Figure 92 shows some examples of bass pedals by alternating notes. In panel *a* for keyboard E is the dominant note of the key and the pedal merges into an E major chord. In the violin work, *b*, the pedal continues for $15\frac{1}{2}$ bars to close on a long G major chord forming a cadence. The pedal note is played on the open G string of the instrument, which is the dominant note of the key. Similarly in *c* the pedal bass note is on the open G string. It persists for 16 bars, that is, it is struck 48 times. In *a* and *b* of Figure 92 the pedal note sounds at the off-beat, while in *c* it sounds on the beat.

There are several instances in Bach where the sustained note is not in the bass. The terms

Figure 93: Pedal notes which are not in the bass. a) pedal on open E string of violin in Partita No.3 in E, and b) pedal on open A string in Prelude to Cello Suite No.1 in G.

‘inverted pedal’ and ‘internal pedal’ have been used, but I will not press this distinction. Two examples are given in Figure 93. In the upper panel *a*, from the violin partita in E, BVW 1006, the note E – the tonic of the work – is repeated 86 times in successive bars, mainly on the open E string. In the famous cello prelude at *b* the pedal note A starts at $\frac{3}{4}$ through the piece and is sounded 53 times over successive bars. Remarkably, it is followed immediately by the pedal D in the bass, seen in the bottom stave of Figure 93. This continues for 5 bars up to the final cadence on a G major chord. As in the keyboard prelude in C from WTC I, the two pedals in succession greatly define the character of the piece. Indeed the well know rocking opening to the ’cello prelude, with arpeggios across three strings, has a pedal G sounded 9 times in the first bars. In both Figure 93 a and b the pedal semiquaver is on the off-beat. I will also cite the Prelude to the D major ’cello suite No. 6 which is replete with pedals on the D, E and G, in all cases off the beat (middle note of a triplet), and the famous ‘Raindrop’ prelude in D♭ by Chopin with its ever repeated A♭/g♯ note from which the piece gained its name.

A common pedal-like feature has been shown in Figure 52, §12.1, in the third Handel quotation and the next, from Bach. In both of these stepwise, scale-like movement in one voice is alternated with a static note. The difference from the authentic long pedal is one of degree rather than kind; the static note remains for only a few bars, and the moving voices are scales or stepwise notes rather than unambiguous chords of harmonic significance. In some pieces the one voice moving stepwise is joined by a second moving with it in parallel 3rds, 6ths or 10ths. Some examples from keyboard pieces by Handel and Scarlatti are given in Figure 94. By such means a linear intervallic pattern in two voices could be elaborated with a pedal note. In each the static note is either the tonic or dominant of the prevailing key. Closely related to these are the suspension of the tonic note in the bass a few bars in which 6 4 second inversion chords form above them, as in the Mozart examples of Figure 57c and 58. These too are a small scale type of pedal.

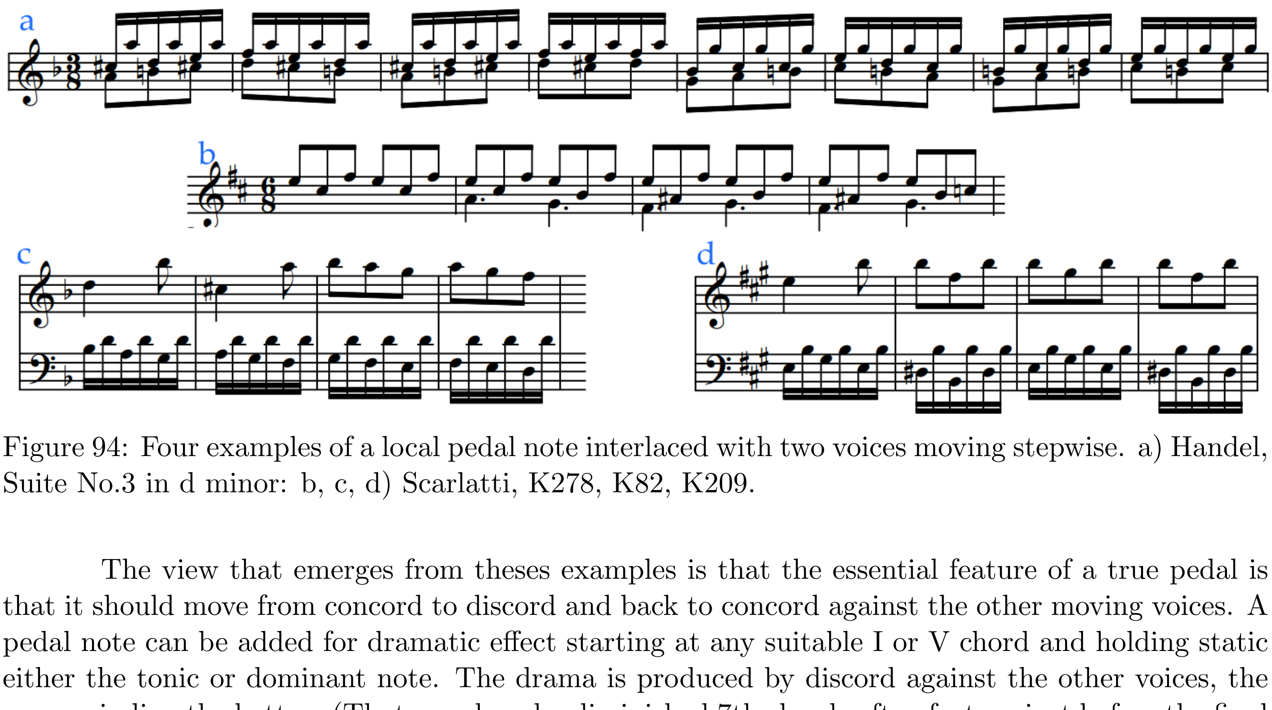

Figure 94: Four examples of a local pedal note interlaced with two voices moving stepwise. a) Handel, Suite No.3 in d minor: b, c, d) Scarlatti, K278, K82, K209.

The view that emerges from theses examples is that the essential feature of a true pedal is that it should move from concord to discord and back to concord against the other moving voices. A pedal note can be added for dramatic effect starting at any suitable I or V chord and holding static either the tonic or dominant note. The drama is produced by discord against the other voices, the more grinding the better. (That may be why diminished 7th chords often feature just before the final cadence.) Though a pedal can be added in the middle of a piece to close on an internal cadence, it is more usual for one to be placed at the very beginning and particularly at the end to merge into the final chord. In the reference key of C, a bass pedal would start as the root C of Ia or G of Va, and continue until that chord again returns. In terms of writing software, the pedal would best be placed in an extra voice since for much of its duration it is not part of the harmony, so cannot be allocated to one of the given 3 or 4 voices. Whether it is sustained as one continuous note or alternated with other voices will depend on the intended style of the piece and the instrument on which it is to be played. Applying these guidelines to the hymn tune of Figure 36, there could be

1. a bass C, two octaves below middle C, held from notes 0 to 3.
2. a high pedal C from notes 5 to 9, picking up the soprano C,
3. a mid range pedal G below middle C from notes 13 to 18 and even possibly to 21, picking up the tenor G.

Bach's practice shows that where the pedal note is in interlaced repeated short notes, it can sound either on or off the beat.

16.5 Compound melody

I gave my opinion in the introduction to this section that the voice-splicing techniques of arpeggiation, octave transfers, pedal notes and compound melody are not in all cases readily distinguishable, since all involve combining notes from two or more given voices, moving with the harmonic rhythm, into a single line. If there are only two given voices, arpeggiation consists merely of alternating between them using a suitable rhythm which matches the time signature. Octave transfers might be used to remove large intervals, and the resulting smaller intervals filled in with passing notes to create a smooth, stepwise single line which we can rightly call a compound melody. To be tuneful, however, it also requires a shape and phrase structure, just as the piece as a whole was given a shape by the voice guide curves in §5.5. Again we turn to Bach and other masters for examples.

Figure 95 quotes a few bars from Bach's double violin concerto in d minor, a truly great work. At *a*, which runs onto a second line, is what Bach actually wrote. At *b* I have followed Walter Piston's analysis[17] to show how it can equally well be written as two distinct lines and indeed played as such on two separated instruments. The appearance of notes in green indicates the introduction of a third voice at lower pitch. At *c* I have removed the octave jumps from the first two bars, and at *d* I have removed the passing notes and suspensions to reduce it to two voices moving in parallel thirds. Stripped down to this, it looks much like a linear intervallic pattern.

The reverse process of converting 2-voice writing into a single compound line is demonstrated in Figure 96. Here the top line is the right hand part of the Allemande to Bach's d minor French Suite, and the lower stave is my translation of this into one voice which, after some upward transposition, could be played on a violin. The same can be done for several bars of the Allemande to the fourth (E♭) French Suite, the Prelude in f minor, WTC Book I, and no doubt several other of Bach's works, such is his contrapuntal mindset. Nor is the practice confined to Bach; there are ample instances of 2-voice writing in Handel's instrumental works which could be compounded to a single voice.

From looking at several movements by Bach I come to the view that he writes compound melody with two and three voices, but rarely four. The number of discernible voices will change through the piece. Similarly, his arpeggios are often across only three notes, except where a long arpeggio on one triad fills a whole bar. Weaving from note to note of a chord can hold the listener's interest for longer than just playing and holding the one base chord, so arpeggiation and compound

[17] 'Counterpoint', publ. Victor Gollancz, 1984.

Figure 95: Extract from the Violin 2 part of Bach's double concert in d minor BWV 1043, first movement, showing compound melody in two then three voices.

melody are tools for extending a chord over several beats. Indeed, it is remarkable how Bach can sustain interest on only one triad for many bars using arpeggios, scale segments and tonic or dominant pedals. A prime example is the opening of the 3rd solo violin partita in E, where the E major harmony lasts for at least the first 16 bars.

Figure 96: The first few bars of Bach's Allemande from his French Suite No 1, (right hand only) showing in the lower stave how it readily condenses into a compound line for one voice.

17 Program 12: algorithms and examples

Program 12 is intended to create some of the forms of decorations described in §16 above which is derived from two or more voices. I will start with arpeggiation, from which the other processes derive.

1. A 3-dimensional array A is defined to hold the new arpeggiated voice line. One dimension A_1 stores the pitch at each position, counting from the beginning at 0. A_2 stores the duration of the new note, and A_3 stores for reference the number of the chord in the given starting sequence from which the new note derives. A is initialised so that its durations are identical with those of the given hymn tune.

2. Arpeggiation is carried out by a procedure (subprogram) to which parameters are passed from the main program for each chord of the hymn tune. These parameters give the chord's number k counting from 0, the number of notes N_k into which the chord is to be divided, and the order of the voices to be used for each chord. The longest duration catered for is one whole bar of 12/8 time, so a list of up to 12 voice selections can be specified, the rest of the 12 being dummy values.

3. The arpeggiation procedure works in much the same way as the LIP procedure in Program 10. For each given chord at position k the N_k pitches and durations of its arpeggio are written to a dummy array D, the array A to the right of k is shifted up by $N - 1$ positions, and then D is copied into A starting from k.

Figure 93 presents a variant of the hymn tune in Figure 36 with one linear intervallic pattern inserted. In the staves below is an arpeggiation produced by Program 12. I have taken the view that arpeggios are generally most comfortable to the ear when the notes are mostly of equal length, though some dotted rhythms can prevent monotony. In Figure 93 the equal and dotted rhythms have been selected at random by the software using random numbers. In Figure 94 I give the first few bars broken into even triplets. Clearly other rhythms could be applied. For the arpeggiation in Figure 93 the order of notes has been set as SABTAS. All 6 notes are used for a dotted minim, the first 4 for a minim and only the first 3 for a crotchet. Clearly another reasonable choice of rhythms would be to include all 4 voices of each chord by breaking each crotchet into semiquavers. Note how from bar 11 the rest in the bass during the LIP has been converted into dotted quaver and semiquaver rests in the arpeggio, so adding welcome melodic interest. For Figure 98 in 9/8 time the voice order is SAB TAS BTS, with all 9 being used for a dotted minim, 6 for a minim and 3 for a crotchet. In both examples the voice order is maintained throughout the piece. It could readily have been varied or even replaced by a random order, but I have taken the view that something of a recognisable pattern is desirable for the sake of musical sense. Musically, neither of these two arpeggiations is very intelligible when listened to alone, mainly because of the large leaps, but when heard as an accompaniment to the hymn tune, each provides useful rhythmic interest and movement.

We should expect that moving some of the arpeggiated notes in Figures 97, 98 in pitch would remove the jerkiness and give a more musical line. The algorithm in Program 12 which effects octave transfers computes the running mean M_k of the pitches over 5 adjacent notes, 2 either side of the note in question (but ignoring rests):

$$M_k = \frac{1}{5} \sum_{j=-2}^{2} P_{k+j}$$

where P_k is the pitch on the diatonic scale. The absolute pitch deviation $|P_k - M_k|$ of the central note at position k from the mean is compared with a threshold value, t, and the note moved towards

Figure 97: Arpeggiation of 4-part harmony into a random mixture of dotted and equal quavers.

the mean M_k if t is exceeded. The shift is by 7 diatonic steps, which is one octave. The running mean progresses from the beginning of the piece (using a suitable mean for the first two and last two notes), and therefore changes made to notes at positions $< k$ of the central note are taken up in calculating the local running mean at k.

Figure 99 shows the effect of changing the threshold on the first few bars of a arpeggiated line. The original, written in the alto clef, is the top line in this figure. Making no change is equivalent of having an infinite threshold, but in practice this matches $t = 7$. If $t < 3 \cdot 5$ the voice line becomes inverted as can be seen for $t = 2$ in the bottom stave. All modified lines are distinct improvements on the original, though at the cost of the voice lines becoming tangled. For instance, the bass note may be moved up so that it is no longer the lowest, or the soprano no longer the highest. This folding over of the higher and lower notes means that we lose the shape of the voice lines imposed by the guide curves in Program 5. This argues that instead of making octave transfers to an arpeggiation derived from widely spaced guide curves, it would be better to place the guide curves close together in the first place. I have found, however, that unless the guide curve are sufficiently far apart, the chords generated will be missing some notes. Though Program 7 does act to add the missing notes

Figure 98: First line of the same hymn tune (from Figure 57) arpeggiated in compound time.

so that each triad is complete, the shape of the guide curve can be compromised. This all means that the purpose of the arpeggiated line needs to be clear, so that meaningful musical choices are made. For instance, after octave transfers it could be a decorative moving line sandwiched between the original chorale soprano and base, as in a chorale prelude for keyboard. This would at least have the merit of retaining the intended bass and soprano lines.

Octave transfer can be combined with arpeggiation to produce arpeggiated chords, as in the example of Figure 100, derived from the same given chords sequence. Here the lower line was produced without octave transfer using the voice order ATBTAT, while the upper was with octave transfer, $t = 4 \cdot 2$, with voice order SASTAS. I then moved about eight notes through an octave by hand to remove some of the larger leaps in the bass and to remove intervals of a 4th or 8ve. I do not consider such by-hand adjustment as 'cheating' . Indeed the algorithms do not implement all forms of arpeggiation or octave transfer which one might like to carry out, such as the sporadic jump of 7th or octave as seen in Bach's double violin concerto, Figure 95. These are really a matter of musical taste and personality. They are probably best decided upon by running the program with two or three different thresholds t, and choosing by hand which seem best in the context. All programs in the suite are just tools to be used with taste and intelligence.

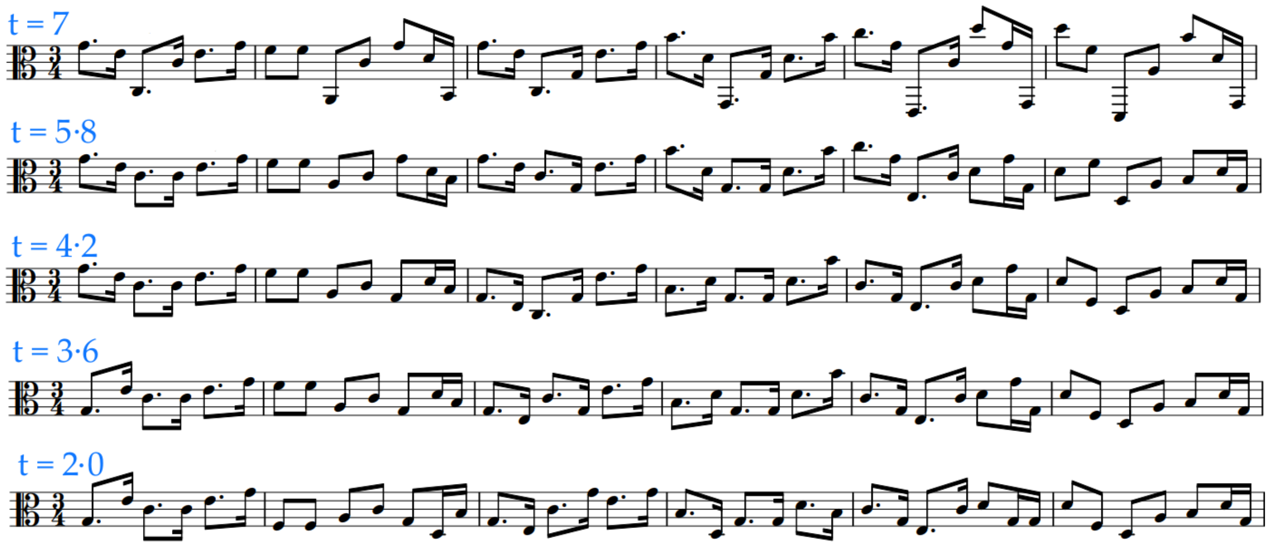

Figure 99: The effect of changing the threshold t above which notes are moved 1 octave towards the running mean pitch.

Figure 100: 2-note chords produced by combining two arpeggiations of the same given chords, with different voice orders.

Closing remarks

I have attempted with these 12 computer programs to capture some of the elementary theory and practice of western music. They are all based on the premise that short pieces and short sections of longer pieces can be created by first forming a meaningful progression of 4-voice chords, like a hymn tune, extending it with linear intervallic patterns, and decorating the result with passing, auxiliary and other non-harmony notes. Having written the programs, I confess to not having spent much time in using them, like a tool box, to create short pieces of music. Perhaps when time allows...!

The examples quoted from great composers, illustrating the many techniques and devices in musical invention, rightly make us in awe of their achievement in writing music of real quality. Very little in the programs and in the commentary on them in this article touches on the development of music themes, much less the large scale organisation of material into an integral extended composition. However, a study of the building blocks of music should give us greater insight into and appreciation of master works.

John Coffey, Cheshire, England, 2020
with modest editorial changes May 2024 and May 2026.

Appendix 1:
Note nomenclature, file conversion & Program 3

Nomenclature of notes and chords

Several sets of symbols are used within the programs to denote single notes and triadic chords.

1. Roman numerals nomenclature (Figure 1) for diatonic triads relative to the tonic, which is labelled I in major or i in minor. Upper case denotes a major triad, lower case a minor one. Throughout the programs the reference key is C: either C major or c minor.

2. In the simpler programs which deal only with the major mode, single notes are coded upwards from the lowest string on the viola, C3. This is note 0, and 1 is added for each semitone higher, making Middle C code 12.

3. The internal coding in item 2 is closely related to the MIDI scale in which Middle C is labelled 60, and each semitone rise in pitch adds 1. (There are 12 semitones per octave.)

4. Programs which deal with the minor mode and with modulation require a label for each triad of the chromatic scale, plus diminished 7th chords. These use

   - 1 for C major, 2 for C♯ or D♭ major, 3 for D major, and so on up to 12 for B major, then
   - 13 for c minor, 14 for c♯ minor, and so on to 24 for b minor, then
   - 25 for the 6-4 (second inversion) chords I_c, 26 for V_c, 27 for ♭III_c, then
   - 28, 29, 30 for the three distinct diminished 7th chords containing respectively the notes C, C♯ and D. I refer to this latter nomenclature as 'Table 10 notation', following §5.

5.

Program 3 and Lilypond score engraving

It is necessary to present the music created by Program 2 and indeed by all programs in this suite as sheet music and a musical sounds. For this I have used the excellent Lilypond program, a free open source powerful music engraving software from *www.lilypond.org*. It is accompanied by editing software called Frescobaldi. Lilypond has its own internal coding for notating music. It gives as output both the engraved score and a MIDI file which can be played on the computer or any MIDI instrument. I first wrote Program 3 to read the output text file from Program 2, in which pitch and duration are coded according to my own format, and convert it into a Lilypond file. As the suite of programs grew, variants of Program 3 have been developed to make other conversion, so now 'Program 3' has become as suite of essential file conversion programs. Its variants are:

1. Program 3a : converts text files in my coding to Lilypond files, in up to 4 voices, one note per voice. It accommodates dotted notes and triplets,

2. Program 3b : converts text files from programs which output chords into Lilypond files,

3. Program 3c : convert in the reverse direction, taking a Lilypond file and outputting DATA statements (in the BASIC computer language) in the internal representation used within my suite of programs. Currently such DATA statement are use to input to other programs, rather than reading an external file.

Many figures in this article have been produced by programs in the suite, converted by Program 3 and engraved by Lilypond, corrected by hand using Frescobaldi, and converted back to internal representation for input into another program in the suite. The input and output formats of the several programs have been arranged to facilitate the transfer of musical data from one program to another.

Appendix 2:
Recognising key and time signatures

: Program 0 In practical music exams at the elementary level the candidate is asked to identify the time signature of the short piece played by the examiner and to say whether it is in the major or minor mode. A kind examiner would emphasise the bass so that a clear left-right 2-in-a-bar is heard, or the equivalent for 3- and 4-time. Major or minor is recognised mainly by the flattened third of the scale which gives a sad feeling to the sound. I have considered what would be involved in carrying out this recognition by computer on the soprano line only of a piece, no bass.

Even though any child with a musical ear can recognised both the pulse and the mode of the piece, I have found it quite subtle to devise an algorithm for the general case. The algorithm also recognises the key, not just the mode, from a data statement giving the pitch and relative duration of successive notes along the vocal line. The key is a feature of the piece of music as a whole. The most obvious guide to key is to hear the closing cadence; in almost all cases the last note is the tonic. However, this in itself does not distinguish major from minor. If we are given only a part of the piece, with the last notes missing, there is likely to be ambiguity about the key and also the mode. For example, many pieces are in binary A-B form where the first A section closes on the dominant chord of the key, so if we were given only this A section, we might conclude that it is in the dominant key. Similarly, many pieces in a minor key have extensive sections in the relative major. Also baroque pieces in predominantly the minor would close on the major triad, the *tierce de picardie.*

I have found recognising the time signature even more of a problem than finding the key. While the bass may have a steady pulse and be unambiguously 2, 3, 4 or even 5 in a bar, the melody will usually be broken by decorative shorter notes and possibly also tied notes. One clue in the melody is to find sequences or otherwise repeated snatches of rhythm which are probably one of two bars in length. The implied harmonic chord often changes at the beginning of a bar, so chord changes are also a clue.

'Program 0' works broadly as follows. A data statement is provided which lists the notes of the melody in pitch and relative duration. In principle these could be converted from a given MIDI file, though I have not attempted to write such a conversion program. The analysis is in three stages.

Stage 1: The first stage eliminates most of the 12 major and 12 minor keys on the grounds that the melody does not contain sufficient notes in their scale. This is done by providing for each major and each minor key a list of notes in their scale. The internal notation is C = 0, C♯ = 1, D = 2, and so on to B = 11. Thus C major has the notes (0, 2, 4, 5, 7, 9, 11) mod 12, and c minor has (0, 2, 3, 5, 7, 8, 9, 10, 11) mod 12. The list for the minor contains notes of both the ascending and descending forms. Each note of the melody is reduced mod 12 and compared against these 12 lists for major and 12 for minor. For each note in the scale of key X its duration is added to a running total for X. The total duration of notes spent in each key is sorted in order to rank the keys. There may be one key which clearly scores much higher than others, but usually I find that one or two major keys and one, two or three minor all score about the same. So the search has been reduced to selected one key from between 2 to 6 candidates.

Stage 2: Various tests are made on these higher scoring keys and their results compounded with empirically determined weightings into a single overall rating, R. It is necessary to know whether the whole piece (or self-contained section of a longer piece) is given because much weight is given to the closing cadence. Tests and weightings are:

1. checking whether the first and last notes are the tonic,
2. if the last note is the tonic, checking whether the penultimate is the 2nd, 5th or leading note of the scale, as D → C or as G → C in C major or minor,
3. counting the number of leading notes (e.g. B in C major or c minor),
4. counting the number of times the leading note moves to the tonic, as B → C.
5. counting the number of steps in a descending minor scale, as B♭ → A♭ and A♭ → G in c minor,
6. counting the number of major and minor third degrees of the scale.

To determine key irrespective of mode, most weight is given to the last note being the tonic, and to the leading note rising to the tonic. To distinguish major from minor weight is giving to the number of minor or major third scale degrees, and a lesser weight to the descending minor scale.

I find that the key is identified correctly in most cases, but a few remain where it could be the relative or the other mode. Despite these fairly comprehensive tests, therefore, I find that sections of some pieces remain ambiguous between major and its relative minor until further examination is made of the triads implied by adjacent notes in the melody. The harmonic rhythm is important to identifying the key.

Perfecting this algorithm for the most general case remains a work in progress.