Chapter 6
Fourier Series

6.1 **Trigonometric Series** The second most natural way to represent a general function by familiar functions after power series is to try trigonometric series. By a trigonometric series we mean a series of the form

$$\frac{1}{2}a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx)$$

where x is a variable and a_n, b_n are fixed. The mysterious factor 1/2 in front of a_0 will be explained shortly.

Whereas power series are used to solve ordinary differential equations, trigonometric series are used to solve partial differential equations. For example, harmonic analysis arises from trying to solve the wave equation with trigonometric series. The idea originates with Fourier who was concerned with the heat equation.

There is a connection between trigonometric series and power series via the medium of complex numbers. If we write

$$\cos nx = \frac{e^{inx} + e^{-inx}}{2}, \quad \sin nx = \frac{e^{inx} - e^{-inx}}{2i}$$

then we have

$$\frac{1}{2}a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx) = \sum_{-\infty}^\infty c_n e^{inx}$$

for suitable (complex) coefficients c_n. We call the series on the right hand side the **complex exponential form** for the trigonometric series on the left hand side. Observe that it is a power series. (Powers of e^{ix}, including negative powers!) It is possible to develop the theory of Fourier series in terms of complex exponential series of this type. We shall work with trigonometric series, though we may occasionally use the complex exponential form if convenience dictates.

Exercises 1. Find the complex exponential form for the trigonometric series

$$\sum_1^\infty \frac{\sin nx}{n}.$$

2. Find the trigonometric form for the complex exponential series

$$\sum_{-\infty}^\infty \frac{e^{inx}}{1 + in}.$$

6.2 **Uniformly Convergent Trigonometric Series** Suppose that the trigonometric series

$$f(x) = \frac{1}{2}a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx)$$

is uniformly convergent over all real x, and that its sum is $f(x)$. Then, by 4.5, $f(x)$ is continuous for all x and, by 4.7, we can integrate the series term by term over any bounded closed interval $[a, b]$. If we multiply both sides of the above equation by the bounded function $\cos mx$ then the series is still uniformly convergent (see 1.6, Question 5) and if we integrate over the interval $[-\pi, \pi]$ then we can use the identities

$$\begin{aligned} \int_{-\pi}^{\pi} \cos mx \cos nx \, dx &= 0 \quad (m \neq n) \\ &= \pi \quad (m = n \geq 1) \\ &= 2\pi \quad (m = n = 0) \end{aligned}$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0 \quad (\text{all } m, n)$$

to obtain

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos mx \, dx &= \frac{1}{2}a_0 \int_{-\pi}^{\pi} \cos mx \, dx + \sum_1^\infty a_n \int_{-\pi}^{\pi} \cos mx \cos nx \, dx \\ &\qquad + \sum_1^\infty b_n \int_{-\pi}^{\pi} \cos mx \sin nx \, dx \\ &= \pi a_m \end{aligned}$$

and hence, changing to n,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

for all $n \geq 0$. The factor $1/2$ appears in the standard formulation of a trigonometric series in order to produce a formula for a_n which holds for all n including $n = 0$.

Similarly, multiplying through by $\sin mx$ and using the further identities

$$\begin{aligned} \int_{-\pi}^{\pi} \sin mx \sin nx \, dx &= 0 \quad (m \neq n) \\ &= \pi \quad (m = n) \end{aligned}$$

we have

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \sin mx \, dx &= \frac{1}{2}a_0 \int_{-\pi}^{\pi} \sin mx \, dx + \sum_1^\infty a_n \int_{-\pi}^{\pi} \sin mx \cos nx \, dx \\ &\qquad + \sum_1^\infty b_n \int_{-\pi}^{\pi} \sin mx \sin nx \, dx \\ &= \pi b_m \end{aligned}$$

which, on putting $m = n$, gives

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

for all $n \geq 1$.

Exercises Show that the trigonometric series

$$\sum_0^\infty r^n \cos nx, \quad \sum_1^\infty r^n \sin nx$$

are uniformly convergent over all real x provided $0 < r < 1$.

Find their sums.

6.3 **Fourier Series** We have just shown that if a function $f(x)$ can be represented as a trigonometric series

$$f(x) = \frac{1}{2}a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx)$$

then the coefficients a_n, b_n are given by the formulae

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

at least in the case where the series converges uniformly over the interval $-\pi \leq x \leq \pi$.

We call the numbers a_n, b_n defined by the above formulae the **Fourier coefficients** of $f(x)$, and the trigonometric series which has these coefficients the **Fourier series** of $f(x)$.

Any function for which the integrals defining the Fourier coefficients exist has a Fourier series. For example, any continuous function has a Fourier series. More generally, any $f(x)$ integrable in the sense of Riemann (or Lebesgue) over $-\pi \leq x \leq \pi$ has a Fourier series.

For the Fourier series to be any use we must be able to prove that it converges to the generating function $f(x)$ for a wide class of functions, at least including any that might come up in any application we have in mind. Fortunately this can be done and is the business of the remainder of this chapter.

6.4 **Examples of Fourier Series** We leave it as an exercise for the reader to verify that the following functions have the Fourier series indicated. We use the symbol $=$ to mean 'has the Fourier series' here.

$$\begin{aligned}
\frac{x}{2} &= \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \\
\frac{x^2}{4} &= \frac{\pi^2}{12} - \cos x + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \dots \\
\frac{\pi}{4}\operatorname{sgn} x &= \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \\
\frac{\pi}{4}|x| &= \frac{\pi^2}{8} - \cos x - \frac{\cos 3x}{3^2} - \frac{\cos 5x}{5^2} - \dots
\end{aligned}$$

If we allow ourselves the temporary licence of assuming = to mean what it usually means then we can deduce, e.g., putting $x = \pi/2$ in the first series,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

(c.f. 5.7) or, putting $x = \pi$ in the second series,

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

Exercise By substituting suitable values in the above series (and assuming = to be true) show that the following series have the sums indicated.

$$\begin{aligned}
1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots &= \frac{\pi^2}{12} \\
1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots &= \frac{\pi^2}{8}
\end{aligned}$$

6.5 **Periodic Functions** An important feature of the trigonometric functions $\cos nx$, $\sin nx$ is their **periodicity**. They repeat themselves over every interval of length 2π. In general we define a function $f(x)$ to be 2π**-periodic** or to have **period** 2π if

$$f(x + 2\pi) = f(x)$$

for all x. It follows that any trigonometric series and therefore any Fourier series is 2π-periodic.

Whilst it is enough for the function $f(x)$ to be defined only on the interval $-\pi \le x \le \pi$ to define its Fourier series, the usual situation is that $f(x)$ is defined outside this interval too, and moreover is **not** 2π-periodic! But the Fourier series of $f(x)$ if it converges at all must converge to something which **is** 2π-periodic.

Given a function $f(x)$ defined for $-\pi \le x \le \pi$ we shall define its 2π**-periodic extension** $g(x)$ to be the 2π-periodic function defined on the whole real line which is equal to $f(x)$ on the interval $-\pi \leq x \leq \pi$. This may involve some kind of compromise at $\pm\pi$ if $f(\pi) \neq f(-\pi)$. The usual practice is to take $g(\pi) = g(-\pi)$ equal to the average of $f(\pi)$, $f(-\pi)$. The 2π-periodic extension of $f(x)$ will have discontinuities at $x = \pm\pi$ unless $f(\pi) = f(-\pi)$.

In the happy event of a Fourier series converging to its generating function $f(x)$ over the interval $-\pi \leq x \leq \pi$ it will converge to the 2π-periodic extension of $f(x)$ outside this interval. (Which may not be the same as $f(x)$.) The Fourier series only sees $f(x)$ on the interval $-\pi \leq x \leq \pi$ and thinks $f(x)$ is otherwise 2π-periodic.

Exercise Show that if $f(t)$ is continuous and 2π-periodic then

$$\int_{x-\pi}^{x+\pi} f(t)dt = \int_{-\pi}^{\pi} f(t)dt$$

for any x.

6.6 **Behaviour of Fourier coefficients** The main result we shall prove in this section is the Riemann-Lebesgue Lemma which states that the Fourier coefficients of any integrable function must $\to 0$. The result was first proved by Riemann for his kind of integral and later generalised by Lebesgue to his kind of integral. We shall deduce the Riemann-Lebesgue Lemma as a corollary of another result on Fourier coefficients known as Bessel's Inequality. We shall prove Bessel's Inequality for continuous functions though the proof is equally valid for Riemann integrable functions.

Theorem Bessel's Inequality If $f(x)$ is continuous over the interval $-\pi \leq x \leq \pi$ and has Fourier series

$$\frac{1}{2}a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx)$$

then

$$\frac{1}{2}a_0^2 + \sum_1^N (a_n^2 + b_n^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} \big(f(x)\big)^2 dx$$

for all $N \geq 1$.

Proof Observe that

$$
\begin{aligned}
0 &\le \frac{1}{\pi}\int_{-\pi}^{\pi}\left(f(x) - \frac{1}{2}a_0 - \sum_{1}^{N}(a_n\cos nx + b_n\sin nx)\right)^2 dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\big(f(x)\big)^2 dx - \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\left(\frac{1}{2}a_0 + \sum_{1}^{\infty}(a_n\cos nx + b_n\sin nx)\right) dx \\
&\qquad + \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}a_0 + \sum_{1}^{\infty}(a_n\cos nx + b_n\sin nx)\right)^2 dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\big(f(x)\big)^2 dx - \frac{1}{2}a_0^2 - \sum_{1}^{N}(a_n^2 + b_n^2)
\end{aligned}
$$

which gives the required result. Q.E.D.

Corollary 1 The series

$$\sum_{1}^{\infty}(a_n^2 + b_n^2)$$

converges.

Proof It is a series of positive terms whose partial sums are bounded. Q.E.D.

Corollary 2 Riemann-Lebesgue Lemma If $f(x)$ is continuous over the interval $-\pi \le x \le \pi$ and has Fourier series

$$\frac{1}{2}a_0 + \sum_{1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

then $a_n \to 0$, $b_n \to 0$ as $n \to \infty$.

Proof The result follows immediately from the fact that the series

$$\sum_{1}^{\infty} a_n^2, \quad \sum_{1}^{\infty} b_n^2$$

both converge. Q.E.D.

Exercises Show that if $f(x)$ has a continuous derivative on $-\pi \le x \le \pi$ and $f(\pi) = f(-\pi)$ then

(i) $na_n \to 0$, $nb_n \to 0$ as $n \to \infty$,

(ii) the series

$$\sum_{1}^{\infty}\big(|a_n| + |b_n|\big)$$

converges.

Hints (i) Consider Fourier coefficients of $f'(x)$.
(ii) Use Cauchy's Inequality

$$\left(\sum_1^N x_n y_n\right)^2 \leq \sum_1^N x_n^2 \sum_1^N y_n^2$$

for any real x_n, y_n.

6.7 **Dirichlet's Kernel** Let $f(x)$ be continuous over $-\pi \leq x \leq \pi$ and let

$$s_N(x) = \frac{1}{2}a_0 + \sum_1^N (a_n \cos nx + b_n \sin nx)$$

be the Nth partial sum of the Fourier series of $f(x)$. In this section we derive an elegant and useful formula for $s_N(x)$ known as Dirichlet's Formula.

By the definition of a_n, b_n (see 6.3) we have

$$\begin{aligned} s_N(x) &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2} + \sum_1^N (\cos nx \cos nt + \sin nx \sin nt)\right) f(t)dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2} + \sum_1^N \cos n(x-t)\right) f(t)dt. \end{aligned}$$

Now for any real x we have

$$\begin{aligned} \sum_1^N \cos nx &= \operatorname{Re} \sum_1^N e^{inx} \\ &= \operatorname{Re} \frac{e^{i(N+1)x} - e^{ix}}{e^{ix} - 1} \\ &= \operatorname{Re} \frac{e^{i(N+1/2)x} - e^{ix/2}}{e^{ix/2} - e^{-ix/2}} \\ &= \frac{1}{2}\frac{\sin(N+1/2)x}{\sin x/2} - \frac{1}{2}. \end{aligned}$$

Therefore we obtain

$$s_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x-t)f(t)dt$$

where

$$D_N(x) = \frac{\sin(N+1/2)x}{\sin x/2}.$$

The function $D_N(x)$ is called **Dirichlet's kernel**. Its graph looks rather like the Forth Bridge.

An important property of the Dirichlet kernel $D_N(x)$ which we shall frequently make use of in the next few pages is the following.

Lemma For all $N \geq 0$ we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin(N+1/2)x}{\sin x/2}dx = 1.$$

Proof From the above analysis we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin(N+1/2)x}{\sin x/2}dx &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}+\sum_{1}^{N}\cos nx\right)dx \\ &= 1\end{aligned}$$

for all $N \geq 0$. Q.E.D.

6.8 **Convolutions** The above representation of the Nth partial sum of a Fourier series as an integral is an example of a convolution. The general definition is as follows.

Definition If $f(x)$, $g(x)$ are continuous 2π-periodic functions (defined for all real x) then we define their **convolution** $h(x)$ to be the function

$$\begin{aligned}h(x) &= \int_{-\pi}^{\pi} f(x-t)g(t)dt \\ &= \int_{-\pi}^{\pi} f(t)g(x-t)dt.\end{aligned}$$

To see why the two integrals are equal observe that if we make the substitution $u = x - t$ then we have

$$\begin{aligned}\int_{-\pi}^{\pi} f(x-t)g(t)dt &= \int_{x-\pi}^{x+\pi} f(u)g(x-u)du \\ &= \int_{\pi}^{\pi} f(u)g(x-u)du \\ &= \int_{-\pi}^{\pi} f(t)g(x-t)dt\end{aligned}$$

since the integrand is 2π-periodic. (See Exercise at end of section 6.5.)

6.9 **Pointwise Convergence of Fourier Series** We consider first continuous functions. It can be shown that there exist continuous functions with divergent Fourier series. (See Appendix.) It follows that to prove convergence we must assume more about the function than mere continuity. We show in this section that the Fourier series converges to the function at any point where the function is differentiable.

Theorem If $f(x)$ is continuous and 2π-periodic and if $f(x)$ is differentiable at x then the Fourier series of $f(x)$ converges to $f(x)$ at x.

Proof We have to show (see 6.7) that

$$s_N(x) \to f(x)$$

as $N \to \infty$. We have

$$\begin{aligned} s_N(x) - f(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(N+1/2)t}{\sin t/2} f(x-t)dt - f(x) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(N+1/2)t}{\sin t/2} \big(f(x-t) - f(x)\big)dt, \end{aligned}$$

by the Lemma of 6.7,

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(N+1/2)t \, g(t)dt$$

where

$$g(t) = \frac{f(x-t) - f(x)}{\sin t/2}.$$

Observe that $g(t)$ is continuous for all t. The continuity at $t = 0$ follows from the fact that

$$g(t) = \frac{f(x-t) - f(x)}{t} \, \frac{t}{\sin t/2} \to -2f'(x)$$

as $t \to 0$. Therefore

$$\begin{aligned} s_N(x) - f(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(N+1/2)t \, g(t)dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\sin Nt \cos t/2 + \cos Nt \sin t/2) g(t)dt \\ &= \alpha_N + \beta_N \end{aligned}$$

where α_N is the Nth Fourier cosine coefficient of the continuous function

$$\frac{g(t)\sin t/2}{2}$$

and β_N is the Nth Fourier sine coefficient of the continuous function

$$\frac{g(t)\cos t/2}{2}.$$

Hence $s_N(x) \to f(x)$ as $N \to \infty$ by the Riemann-Lebesgue Lemma. (See 6.6.) Q.E.D.

6.10 **Fourier Series of Piecewise Continuous Functions** We say that a function $f(x)$ has a **jump discontinuity** at x if the limits

$$f(x+0) = \lim_{h\to 0+} f(x+h), \quad f(x-0) = \lim_{h\to 0+} f(x-h)$$

both exist (finite) but are not equal. A **piecewise** continuous function is one which is continuous except at a finite number of points where it has jump discontinuities. For 2π-periodic functions defined over the whole real line we require a finite number of jump discontinuities in any interval of length 2π.

All the theory we have presented so far is valid for piecewise continuous functions. They certainly have Fourier series and the Riemann-Lebesgue Lemma holds. Also the Theorem of 6.9.

A jump discontinuity presents a Fourier series with a dilemma. Which limiting value should it converge to? We shall see that the Fourier series comes up with the happy compromise of converging to the average of the limiting values

$$\frac{f(x+0)+f(x-0)}{2}.$$

Though we have to make some assumptions about differentiability to get a general theorem.

We say $f(x)$ is **piecewise** continuously differentiable if $f(x)$ is continuously differentiable except at a finite number of points where either $f(x)$ or $f'(x)$ (or both) have a jump discontinuity. This is equivalent to demanding that all four limits

$$\begin{aligned} f(x+0) &= \lim_{h\to 0+} f(x+h), & f(x-0) &= \lim_{h\to 0+} f(x-h), \\ f'(x+0) &= \lim_{h\to 0+} f'(x+h), & f'(x-0) &= \lim_{h\to 0+} f'(x-h) \end{aligned}$$

exist everywhere, and that each left hand limit should equal its corresponding right hand limit everywhere except at a finite number of points. For example, the 2π-periodic extension of $f(x) = x$ ($|x| \le \pi$) has jump discontinuities at $x = \pm\pi$, whilst the 2π-periodic extension of $f(x) = x^2$ ($|x| \le \pi$) has jump discontinuities in its derivative at $x = \pm\pi$.

Theorem If $f(x)$ is 2π-periodic and is piecewise continuously differentiable then the Fourier series of $f(x)$ converges to

$$\frac{f(x+0)+f(x-0)}{2}$$

at every x.

N.B. This implies that the Fourier series converges to $f(x)$ wherever $f(x)$ is continuous.

Proof We can assume without any loss of generality that

$$f(x) = \frac{f(x+0) + f(x-0)}{2}$$

for every x, since altering the value of $f(x)$ at a finite number of points will not affect its Fourier coefficients. With this assumption we simply have to show that the Fourier series of $f(x)$ converges to $f(x)$ at every x.

At any point where $f(x)$ is differentiable the theorem of 6.9 (generalised to piecewise continuous functions) shows the Fourier series converges to $f(x)$. At any point where $f(x)$ is continuous the proof of theorem 6.9 is still valid even if $f'(x+0) \neq f'(x-0)$ since the function $g(t)$ defined in the proof of theorem 6.9 is piecewise continuous.

The only outstanding case is where x is a jump discontinuity of $f(x)$. For any such x we have

$$\begin{aligned} s_N(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(N+1/2)t}{\sin t/2} f(x-t)dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(N+1/2)t}{\sin t/2} f(x+t)dt, \end{aligned}$$

(putting $t = -t$)

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(N+1/2)t}{\sin t/2} \, \frac{f(x+t) + f(x-t)}{2} dt$$

(adding and halving)

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(N+1/2)t}{\sin t/2} g(t)dt$$

where

$$g(t) = \frac{f(x+t) + f(x-t)}{2}.$$

This function $g(t)$ is continuous at $t = 0$ since

$$\lim_{t \to 0+} g(t) = \lim_{t \to 0-} g(t) = \frac{f(x+0) + f(x-0)}{2} = f(x)$$

by assumption. Therefore by the previous paragraph the Fourier series of $g(t)$ must converge to $g(t)$ at $t = 0$. Hence $s_N(x) \to f(x)$ as required. Q.E.D.

Corollary If $f(x)$ is differentiable over $|x| \leq \pi$ then the Fourier series of $f(x)$ converges to

$$\frac{f(\pi) + f(-\pi)}{2}$$

at $x = \pm\pi$.

Exercise Find the Fourier series of e^x. Use it to show

$$\sum_{-\infty}^{\infty} \frac{1}{n^2+1} = \pi \coth \pi.$$

6.11 **Uniform Convergence of Fourier Series** We saw in Section 6.10 that a piecewise continuously differentiable function $f(x)$ has a pointwise convergent Fourier series with sum $f(x)$ subject to the condition that

$$f(x) = \frac{f(x+0) + f(x-0)}{2}$$

at discontinuities. We now show that in the case where $f(x)$ (still piecewise continuously differentiable) is continuous everywhere and has $f(\pi) = f(-\pi)$ the Fourier series actually converges uniformly over $|x| \leq \pi$.

Theorem If $f(x)$ is continuous on $|x| \leq \pi$ with piecewise continuous derivative over $|x| \leq \pi$ and if $f(\pi) = f(-\pi)$ then the Fourier series of $f(x)$ is uniformly absolutely convergent over $|x| \leq \pi$.

Proof Suppose $f(x)$, $f'(x)$ have Fourier series

$$\frac{1}{2}a_0 + \sum_{1}^{\infty}(a_n \cos nx + b_n \sin nx),$$
$$\frac{1}{2}a_0' + \sum_{1}^{\infty}(a_n' \cos nx + b_n' \sin nx)$$

respectively. Then, since it is legitimate to integrate $f'(x)$ by parts in this situation, we have

$$\begin{aligned} a_n' &= \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x) \cos nx \, dx \\ &= \frac{1}{\pi} f(x) \cos nx \Big|_{-\pi}^{\pi} + \frac{n}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx \\ &= nb_n, \end{aligned}$$

since $f(\pi) = f(-\pi)$, and

$$\begin{aligned} b_n' &= \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x) \sin nx \, dx \\ &= \frac{1}{\pi} f(x) \sin nx \Big|_{-\pi}^{\pi} - \frac{n}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ &= -na_n. \end{aligned}$$

Therefore by Cauchy's inequality we have, for all $N \geq 1$,

$$\begin{aligned}\sum_1^N |a_n| &= \sum_1^N \left|\frac{b'_n}{n}\right| \\ &\leq \sqrt{\sum_1^N (b'_n)^2 \sum_1^N \frac{1}{n^2}} \\ &\leq \sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} \left(f'(x)\right)^2 dx \sum_1^\infty \frac{1}{n^2}}\end{aligned}$$

by Bessel's Inequality. (See 6.6.) It follows that

$$\sum_1^\infty |a_n|$$

converges, and a similar argument shows that

$$\sum_1^\infty |b_n|$$

converges. Hence the Fourier series of $f(x)$ converges uniformly absolutely over $|x| \leq \pi$ by the Weierstrass M-Test. (See 4.4.) Q.E.D.

Exercise Discuss the uniform convergence of the Fourier series listed in Section 6.4.

6.12 **Cesaro Convergence** Even though the Fourier series of a general continuous function may diverge there are other senses in which convergence can be proved. We shall consider Cesaro Convergence and Mean Square Convergence.

Definition The sequence x_n is said to converge to x in the **sense of Cesaro** if the sequence of averages

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \to x$$

as $n \to \infty$.

Lemma If $x_n \to x$ in the ordinary sense then also $x_n \to x$ in the sense of Cesaro.

Proof Assume first that $x = 0$. Suppose $\epsilon > 0$ is given. Then we can choose N such that $|x_n| < \epsilon$ for all $n > N$. Therefore

$$\begin{aligned}\frac{x_1 + \ldots + x_n}{n} &= \frac{x_1 + \ldots + x_N}{n} + \frac{x_{N+1} + \ldots + x_n}{n} \\ &= A_n + B_n\end{aligned}$$

if $n > N$. The first term $A_n \to 0$ as $n \to \infty$, and the second term

$$\begin{aligned}|B_n| &\leq \frac{|x_{N+1}| + \ldots + |x_n|}{n} \\ &< \frac{\epsilon + \ldots + \epsilon}{n} \\ &= \frac{\epsilon(n-N)}{n} \\ &< \epsilon\end{aligned}$$

for all $n > N$. Therefore

$$\frac{x_1 + \ldots + x_n}{n} \to 0$$

as $n \to \infty$.

If $x \neq 0$ then

$$\frac{x_1 + \ldots + x_n}{n} - x = \frac{(x_1 - x) + \ldots + (x_n - x)}{n} \to 0$$

since $x_n - x \to 0$. Q.E.D.

The converse of the above Lemma is false. There exist sequences which converge in the sense of Cesaro but diverge in the ordinary sense. For example if $x_n = (-1)^n$ then we have

$$\begin{aligned}\frac{x_1 + \ldots + x_n}{n} &= -1/n \quad (n \text{ odd}) \\ &= 0 \qquad (n \text{ even})\end{aligned}$$

therefore $(-1)^n \to 0$ in the sense of Cesaro whilst diverging in the ordinary sense.

Definition The series

$$\sum_1^\infty x_n$$

converges in the **sense of Cesaro** if the sequence

$$s_N = \sum_1^N x_n$$

of partial sums converges in the sense of Cesaro.

Exercise Show that the series

$$\sum_1^\infty (-1)^n$$

converges in the sense of Cesaro and find its Cesaro sum.

6.13 **Uniform Continuity** Our next objective is to prove Fejer's Theorem which states that the Fourier series of any continuous function $f(x)$ with $f(\pi) = f(-\pi)$ converges uniformly to $f(x)$ in the sense of Cesaro. Before we present Fejer's Theorem, however, we must say a few words about uniform continuity.

Definition If I is an interval and $f(x)$ is a function defined on I then we say $f(x)$ is **uniformly continuous** on I if given any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon$$

for all $x, y \in I$ satisfying $|x - y| < \delta$.

N.B. This condition is stronger than ordinary continuity which requires that for each $x \in I$ and $\epsilon > 0$ there should exist $\delta > 0$ **which may depend on** x such that

$$|f(x) - f(y)| < \epsilon$$

for all $y \in I$ satisfying $|x - y| < \delta$. Examples of functions which are continuous but not uniformly continuous are either unbounded, e.g., $f(x) = 1/x$ on the interval $x > 0$, or oscillatory, e.g., $f(x) = \sin(1/x)$ again on the interval $x > 0$.

Theorem If $f(x)$ is continuous on the **closed** interval $a \leq x \leq b$ then $f(x)$ is uniformly continuous on this interval.

Observe That for a closed bounded interval we are able to argue from the weaker concept to the stronger.

Proof Suppose $\epsilon > 0$ is given. Let x_1 be the first point in the interval $[a, b]$ (if there is one) such that

$$|f(x_1) - f(a)| = \epsilon.$$

Let x_2 be the first point in the interval $[x_1, b]$ (if there is one) such that

$$|f(x_2) - f(x_1)| = \epsilon.$$

Let $x_3, x_4, \ldots, x_n, \ldots$ be defined similarly.

At some stage it must be impossible to construct x_n, since otherwise we obtain an infinite sequence

$$a < x_1 < x_2 < \ldots < x_n < \ldots < b$$

such that

$$|f(x_n) - f(x_{n-1})| = \epsilon$$

for all n. Since x_n increases and is bounded above by b there must be a limit $x_n \to x \leq b$ as $n \to \infty$. But therefore also $f(x_n) \to f(x)$ by the continuity which contradicts the fact that $f(x_n)$ is not a Cauchy sequence.

If x_n is the last point that can be constructed then take

$$\delta = \min_{0 \le k \le n} (x_{k+1} - x_k)$$

where $x_0 = a$, $x_{n+1} = b$. Then for any x, y in $[a, b]$ satisfying $|x - y| < \delta$ we must have (at worst) x, y in adjacent subintervals

$$x_{k-1} \le x \le x_k \le y \le x_{k+1}$$

for some $1 \le k \le n$. Therefore

$$\begin{aligned} |f(x) - f(y)| &\le |f(x) - f(x_k)| + |f(x_k) - f(y)| \\ &\le 2\epsilon + \epsilon \\ &= 3\epsilon. \end{aligned}$$

Hence $f(x)$ is uniformly continuous over $a \le x \le b$. Q.E.D.

6.14 **Fejer's Theorem** If $f(x)$ is continuous on $|x| \le \pi$ and $f(\pi) = f(-\pi)$ then the Fourier series of $f(x)$ converges to $f(x)$ in the sense of Cesaro uniformly over $|x| \le \pi$.

Proof Let

$$s_N(x) = \frac{1}{2}a_0 + \sum_1^N (a_n \cos nx + b_n \sin nx)$$

be the Nth partial sum of the Fourier series of $f(x)$ and let

$$\sigma_N(x) = \frac{s_0(x) + s_1(x) + \ldots + s_{N-1}(x)}{N}.$$

Then we have to show that $\sigma_N(x) \to f(x)$ uniformly over $|x| \le \pi$.

From Section 6.7 we have

$$s_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(x - t) f(t) dt$$

where

$$D_N(x) = \frac{\sin(N + 1/2)x}{\sin x/2}.$$

Therefore

$$\sigma_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_N(x - t) f(t) dt$$

where

$$\begin{aligned} F_N(x) &= \frac{D_0(x) + D_1(x) + \ldots + D_{N-1}(x)}{N} \\ &= \frac{1}{N} \frac{\sin^2 Nx/2}{\sin^2 x/2}. \end{aligned}$$

The function $F_N(x)$ is called Fejer's kernel. It has properties similar to those of Dirichlet's kernel $D_N(x)$. In particular

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)dx = 1$$

for all $N \geq 0$. The advantage of Fejer's kernel over Dirichlet's is that it is everywhere ≥ 0.

Now consider

$$\begin{aligned}\sigma_N(x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(t)f(x-t)dt - f(x)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(t)\big(f(x-t) - f(x)\big)dt.\end{aligned}$$

Suppose $\epsilon > 0$ is given. Then by the uniform continuity of $f(x)$ over all real x (extending $f(x)$ 2π-periodically) we can choose $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon$$

for all real x, y satisfying $|x - y| < \delta$. Split the range of integration $-\pi \leq x \leq \pi$ into three subintervals

$$\int_{-\pi}^{\pi} = \int_{-\pi}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\pi}.$$

Estimating the middle integral we have

$$\begin{aligned}\left|\frac{1}{2\pi}\int_{-\delta}^{\delta} F_N(t)\big(f(x-t) - f(x)\big)dt\right| &\leq \frac{1}{2\pi}\int_{-\delta}^{\delta} F_N(t)|f(x-t) - f(x)|dt\\ &< \frac{\epsilon}{2\pi}\int_{-\delta}^{\delta} F_N(t)dt\\ &\leq \frac{\epsilon}{2\pi}\int_{-\pi}^{\pi} F_N(t)dt\\ &= \epsilon.\end{aligned}$$

Observe that this part of the argument doesn't work for Dirichlet's kernel.

Estimating the third integral we have

$$\begin{aligned}\left|\frac{1}{2\pi}\int_{\delta}^{\pi} F_N(t)\big(f(x-t) - f(x)\big)dt.\right| &\leq \frac{1}{2\pi}\int_{\delta}^{\pi} F_N(t)\big(|f(x-t)| + |f(x)|\big)dt\\ &\leq \frac{1}{2\pi}\int_{\delta}^{\pi} \frac{1}{N}\,\frac{2M}{\sin^2\delta/2}dt\\ &= \frac{\pi-\delta}{\pi}\,\frac{1}{N}\,\frac{M}{\sin^2\delta/2},\end{aligned}$$

where

$$M = \max_{|x|\leq\pi} |f(x)|,$$

which $\to 0$ as $N \to \infty$ uniformly over $|x| \le \pi$.

Similarly for the first integral.

Hence $\sigma_N(x) \to f(x)$ uniformly over $|x| \le \pi$. Q.E.D.

Far from Fejer's Theorem being a mere intellectual curiosity it turns out to have several important consequences. The following corollaries provide a sample.

Corollary 1 If the Fourier series of a 2π-periodic continuous function converges at any x it must converge to $f(x)$ there.

Proof In this case we have $s_N(x) \to$ a limit as $N \to \infty$. Therefore also $\sigma_N(x) \to$ this limit as $N \to \infty$. But by Fejer's Theorem $\sigma_N(x) \to f(x)$. Q.E.D.

Corollary 2 If $f(x)$, $g(x)$ are 2π-periodic continuous functions with the same Fourier coefficients then $f(x) = g(x)$ for all x.

Proof We must have $f(x) = g(x) =$ the Cesaro sum of the common Fourier series. Q.E.D.

Corollary 3 Given any 2π-periodic continuous function $f(x)$ and $\epsilon > 0$ there exists a trigonometric polynomial $p(x)$ such that

$$|p(x) - f(x)| < \epsilon$$

for all x.

Proof Use $\sigma_N(x)$ for large enough N. Q.E.D.

6.15 **Mean Square Convergence** Another type of convergence which turns out to be particularly well suited to Fourier series is mean square convergence which we now define.

Definition The sequence of functions $f_n(x)$ is said to converge to the limit function $f(x)$ in **mean square** over the interval $a \le x \le b$ if

$$\int_a^b \big(f_n(x) - f(x)\big)^2 dx \to 0$$

as $n \to \infty$.

Of course we must know that this integral exists which it certainly will if all the functions are e.g. piecewise continuous.

Lemma If $f_n(x) \to f(x)$ uniformly over $a \le x \le b$ then also $f_n(x) \to f(x)$ in mean square over $a \le x \le b$.

Proof Suppose $\epsilon > 0$ is given. Then we can choose N such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $n > N$ and all $a \le x \le b$. Therefore for any $n > N$ we also have

$$\int_a^b \big(f_n(x) - f(x)\big)^2 dx < \int_a^b \epsilon^2 dx = (b-a)\epsilon^2$$

which gives the required result. Q.E.D.

The converse of the above lemma is not true. Consider, for example, $f_n(x) = x^n$ which is not uniformly convergent over $0 \le x \le 1$. Observe however that

$$\begin{aligned}\int_0^1 \big(f_n(x)\big)^2 dx &= \int_0^1 x^{2n} dx \\ &= \frac{x^{2n+1}}{2n+1}\Big|_0^1 \\ &= \frac{1}{2n+1} \\ &\to 0\end{aligned}$$

as $n \to \infty$. Hence $f_n(x) \to 0$ in mean square over $0 \le x \le 1$.

Also, given that $f_n(x) \to f(x)$ pointwise over an interval, it does not necessarily follow that $f_n(x) \to f(x)$ in mean square over the interval. For example, if

$$\begin{aligned}f_n(x) &= x(2-nx) \quad (0 \le x \le 2/n) \\ &= 0 \qquad\qquad (2/n \le x \le 1)\end{aligned}$$

then

$$\int_0^1 \big(f_n(x)\big)^2 dx = \int_0^{2/n} x^2(2-nx)^2 dx = 16/15n^3.$$

Therefore $n^{3/2} f_n(x) \to 0$ pointwise over $0 \le x \le 1$ but not in mean square over $0 \le x \le 1$.

Definition A series

$$\sum_1^\infty f_n(x)$$

is said to converge in **mean square** over the interval $a \le x \le b$ with sum $s(x)$ if

$$\int_a^b \big(s_N(x) - s(x)\big)^2 dx \to 0$$

as $N \to \infty$ where

$$s_N(x) = \sum_1^N f_n(x).$$

Exercise Discuss the mean square convergence of the series

$$\sum_0^\infty x^n$$

over the intervals (i) $-1 \le x \le 0$, (ii) $0 \le x \le 1$.

6.16 **Best Approximation in Mean Square** The integral

$$\int_a^b \big(f(x) - g(x)\big)^2 dx$$

gives a measure of how close the functions $f(x)$, $g(x)$ are over the interval $a \le x \le b$. We shall show in this section that the Nth partial sum

$$s_N(x) = \frac{1}{2}a_0 + \sum_1^N (a_n \cos nx + b_n \sin nx)$$

of the Fourier series of $f(x)$ is the closest one can get to $f(x)$ in this sense over the interval $-\pi \le x \le \pi$ by trigonometric polynomials of degree $\le N$. Explicitly we have the following theorem.

Theorem If

$$p(x) = \frac{1}{2}\alpha_0 + \sum_1^N (\alpha_n \cos nx + \beta_n \sin nx)$$

is any trigonometric polynomial of degree $\le N$ then

$$\int_{-\pi}^{\pi} \big(p(x) - f(x)\big)^2 dx \ge \int_{-\pi}^{\pi} \big(s_N(x) - f(x)\big)^2 dx$$

where

$$s_N(x) = \frac{1}{2}a_0 + \sum_1^N (a_n \cos nx + b_n \sin nx)$$

is the Nth partial sum of the Fourier series of $f(x)$.

Informally, $s_N(x)$ is the best approximation to $f(x)$ in mean square over $|x| \le \pi$ by trigonometric polynomials of degree $\le n$.

Proof Observe that

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\big(p(x)-f(x)\big)^2dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}\alpha_0+\sum_1^N(\alpha_n\cos nx+\beta_n\sin nx)-f(x)\right)^2 dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}\alpha_0+\sum_1^N(\alpha_n\cos nx+\beta_n\sin nx)\right)^2 dx$$

$$-\frac{2}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}\alpha_0+\sum_1^N(\alpha_n\cos nx+\beta_n\sin nx)\right)f(x)dx$$

$$+\frac{1}{\pi}\int_{-\pi}^{\pi}\big(f(x)\big)^2dx$$

$$=\frac{1}{2}\alpha_0^2+\sum_1^N(\alpha_n^2+\beta_n^2)-\alpha_0a_0-2\sum_1^N(\alpha_na_n+\beta_nb_n)$$

$$+\frac{1}{\pi}\int_{-\pi}^{\pi}\big(f(x)\big)^2dx$$

$$=\frac{1}{2}(\alpha_0-a_0)^2+\sum_1^N\big((\alpha_n-a_n)^2+(\beta_n-b_n)^2\big)$$

$$-\frac{1}{2}a_0^2-\sum_1^N(a_n^2+b_n^2)+\frac{1}{\pi}\int_{-\pi}^{\pi}\big(f(x)\big)^2dx$$

which is minimised when $\alpha_n = a_n$, $\beta_n = b_n$ for all n. Q.E.D.

6.18 **Mean Square Convergence of Fourier Series of Continuous Functions** We show in this section that the Fourier series of any continuous function must converge to the function in mean square. The result is a consequence of Fejer's Theorem (6.14) and the best approximation in mean square property of the Fourier series (6.17).

Theorem If $f(x)$ is continuous over $|x| \le \pi$ with $f(\pi) = f(-\pi)$ and has Fourier series

$$\frac{1}{2}a_0+\sum_1^\infty(a_n\cos nx+b_n\sin nx)$$

then

$$\int_{-\pi}^{\pi}\big(s_N(x)-f(x)\big)^2dx \to 0$$

as $N \to \infty$ where

$$s_N(x)=\frac{1}{2}a_0+\sum_1^N(a_n\cos nx+b_n\sin nx)$$

is the Nth partial sum of the Fourier series of $f(x)$.

Proof If we write

$$\sigma_N(x) = \frac{s_0(x) + \ldots + s_{N-1}(x)}{N}$$

then by Fejer's Theorem (6.14) $\sigma_N(x) \to f(x)$ uniformly over $|x| \leq \pi$. Therefore by 6.15 $\sigma_N(x) \to f(x)$ in mean square over $|x| \leq \pi$. But $\sigma_n(x)$ is a trigonometric polynomial of degree $\leq N$ so by the result of 6.17 we have

$$\int_{-\pi}^{\pi} \big(s_N(x) - f(x)\big)^2 dx \leq \int_{-\pi}^{\pi} \big(\sigma_N(x) - f(x)\big)^2 dx.$$

Hence $s_N(x) \to f(x)$ in mean square over $|x| \leq \pi$. Q.E.D.

6.19 **Mean Square Approximation of Piecewise Continuous Functions** Our next task is to generalise the result of 6.18 to piecewise continuous functions. We shall need the following lemma.

Lemma Given any piecewise continuous $f(x)$ over $|x| \leq \pi$ and $\epsilon > 0$ there exists continuous $g(x)$ over $|x| \leq \pi$ with $g(\pi) = g(-\pi)$ such that

$$\int_{-\pi}^{\pi} \big(f(x) - g(x)\big)^2 dx < \epsilon.$$

Proof We must have $f(x)$ bounded on each of its intervals of continuity and therefore globally bounded, say,

$$|f(x)| \leq M$$

for all $|x| \leq \pi$. Let the discontinuities of $f(x)$ occur at $x_1, \ldots, x_n$. (Including $\pm\pi$ if $f(\pi) \neq f(-\pi)$.) For each $1 \leq k \leq n$ define $g(x)$ on the interval $x_k - \delta \leq x \leq x_k + \delta$ ($\delta > 0$ to be specified shortly) to be linear and equal to $f(x)$ at $x = x_k \pm \delta$. Otherwise take $g(x) = f(x)$. Then we must have

$$\int_{-\pi}^{\pi} \big(g(x) - f(x)\big)^2 dx = \sum_{k=1}^{n} \int_{x_k-\delta}^{x_k+\delta} \big(g(x) - f(x)\big)^2 dx$$

$$\leq \sum_{k=1}^{n} \int_{x_k-\delta}^{x_k+\delta} 4M^2 dx,$$

since $|g(x) - f(x)| \leq 2M$ for all $|x| \leq \pi$,

$$= 8n\delta M^2$$

$$< \epsilon$$

if we take $\delta < 8nM^2$. Q.E.D.

We also need the following inequality.

6.20 **Minkowski's Inequality** For any piecewise continuous $f(x)$, $g(x)$

$$\sqrt{\int_a^b \big(f(x)+g(x)\big)^2 dx} \le \sqrt{\int_a^b \big(f(x)\big)^2 dx} + \sqrt{\int_a^b \big(g(x)\big)^2 dx}.$$

Proof For any real λ we have

$$\begin{aligned} 0 &\le \int_a^b \big(\lambda f(x)+g(x)\big)^2 dx \\ &= \lambda^2 \int_a^b \big(f(x)\big)^2 dx + 2\lambda \int_a^b f(x)g(x)dx + \int_a^b \big(g(x)\big)^2 dx. \end{aligned}$$

Therefore the discriminant of this quadratic polynomial in λ must be ≥ 0, i.e.,

$$\left(\int_a^b f(x)g(x)dx\right)^2 \le \int_a^b \big(f(x)\big)^2 dx \int_a^b \big(g(x)\big)^2 dx.$$

(Schwarz's Inequality.) Now consider

$$\begin{aligned} \int_a^b \big(f(x)+g(x)\big)^2 dx &= \int_a^b \big(f(x)\big)^2 dx + 2\int_a^b f(x)g(x)dx + \int_a^b \big(g(x)\big)^2 dx \\ &\le \int_a^b \big(f(x)\big)^2 dx + 2\sqrt{\int_a^b \big(f(x)\big)^2 dx \int_a^b \big(g(x)\big)^2 dx} + \int_a^b \big(g(x)\big)^2 dx \\ &= \left(\sqrt{\int_a^b \big(f(x)\big)^2 dx} + \sqrt{\int_a^b \big(g(x)\big)^2 dx}\right)^2. \end{aligned}$$

The result follows. Q.E.D.

6.21 **Mean Square Convergence of Fourier Series of Piecewise Continuous Functions**
We can now prove the promised generalisation of the result of 6.18.

Theorem The Fourier series of any piecewise continuous function $f(x)$ converges to $f(x)$ in mean square over $|x| \leq \pi$.

Proof Suppose $\epsilon > 0$ is given. Choose (by 6.19) continuous $g(x)$ with $g(\pi) = g(-\pi)$ such that

$$\int_{-\pi}^{\pi} \big(f(x)-g(x)\big)^2 dx < \epsilon^2.$$

Let the Fourier series of $f(x)$, $g(x)$ be

$$\frac{1}{2}a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx),$$
$$\frac{1}{2}c_0 + \sum_1^\infty (c_n \cos nx + d_n \sin nx)$$

respectively, and let

$$s_N(x) = \frac{1}{2}a_0 + \sum_1^N (a_n \cos nx + b_n \sin nx),$$
$$t_N(x) = \frac{1}{2}c_0 + \sum_1^N (c_n \cos nx + d_n \sin nx)$$

be the Nth partial sums. By Minkowski's Inequality (6.20) we have

$$\begin{aligned}\sqrt{\int_{-\pi}^{\pi} \big(s_N(x) - f(x)\big)^2 dx} &\le \sqrt{\int_{-\pi}^{\pi} \big(s_N(x) - t_N(x)\big)^2 dx} \\ &\quad + \sqrt{\int_{-\pi}^{\pi} \big(t_N(x) - g(x)\big)^2 dx} \\ &\quad + \sqrt{\int_{-\pi}^{\pi} \big(g(x) - f(x)\big)^2 dx} \\ &= A_n + B_n + C_n\end{aligned}$$

say. By choice of $g(x)$ we have $C_n < \epsilon$, and $B_n \to 0$ by the theorem of 6.18. As for A_n, observe that $s_N(x) - t_N(x)$ is the Nth partial sum of the Fourier series of $f(x) - g(x)$, therefore by Bessel's Inequality (6.6) we have

$$\begin{aligned}\int_{-\pi}^{\pi} \big(s_N(x) - t_N(x)\big)^2 dx &\le \int_{-\pi}^{\pi} \big(f(x) - g(x)\big)^2 dx \\ &< \epsilon^2.\end{aligned}$$

Hence $s_N(x) \to f(x)$ in mean square over $|x| \le \pi$. Q.E.D.

6.22 **Parseval's Identity** We conclude this chapter by proving a formula due to Parseval which relates the square sum of the Fourier coefficients of a function to the square integral of the function. It is very useful for summing series. (See the Exercise.)

Theorem For any piecewise continuous function $f(x)$ with Fourier series

$$\frac{1}{2}a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx)$$

we have

$$\frac{1}{2}a_0^2 + \sum_1^\infty (a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} \big(f(x)\big)^2 dx.$$

Proof Writing

$$s_N(x) = \frac{1}{2}a_0 + \sum_1^N (a_n \cos nx + b_n \sin nx)$$

we have from the proof of Bessel's Inequality (see 6.6) that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \big(f(x)\big)^2 dx - \frac{1}{2}a_0^2 - \sum_1^\infty (a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} \big(s_N(x) - f(x)\big)^2 dx$$
$$\to 0$$

as $N \to \infty$ by 6.21. Q.E.D.

Exercise Use the Fourier series listed in 6.4 to sum the following series.

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots = \frac{\pi^4}{90}$$
$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{96}$$

6.23 **Miscellaneous Exercises** 1. Show that if

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$$

converges uniformly over $|x| \leq \pi$ then

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

2. Prove that for any real α not an integer

$$e^{i\alpha x} = \frac{1}{\pi}\sum_{-\infty}^{\infty} \frac{\sin(\alpha - n)\pi}{\alpha - n} e^{inx}$$

for all $|x| < \pi$.

What is the sum of the series at $x = \pm\pi$?

Deduce the expansion

$$\cot \alpha\pi = \frac{1}{\pi}\sum_{-\infty}^{\infty} \frac{1}{\alpha - n} = \frac{1}{\pi}\left(\frac{1}{\alpha} + \sum_1^\infty \frac{2\alpha}{\alpha^2 - n^2}\right)$$

for α not an integer.

3. Show that

$$\frac{\pi}{4}\operatorname{sgn} x = \sum_{1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$$

for all $|x| < \pi$.

4. Show that if

$$s_N(x) = \sum_{1}^{N} \frac{\sin(2n-1)x}{2n-1}$$

then

$$s'_N(x) = \sum_{1}^{N} \cos{(2n-1)x} = \frac{1}{2}\frac{\sin 2Nx}{\sin x}.$$

Deduce that $s_N(x)$ has alternate maxima and minima at $x = n\pi/2N$ for $1 \leq n \leq N$.

5. Show that for $s_N(x)$ of Question 4

$$s_N(x) \to \frac{1}{2}\int_0^{\pi} \frac{\sin x}{x}\, dx > \frac{\pi}{4}$$

as $N \to \infty$ and hence (Gibbs' Phenomenon) that the limiting value of the first maximum of $s_N(x)$ as $N \to \infty$ overshoots the limiting value of $\pi \operatorname{sgn} x/4$ as $x \to 0$ by a factor of

$$\frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{x}\, dx > 1.$$

6. Show that if piecewise continuous $f(x)$ has a discontinuity at $x = \alpha$ with jump J then

$$f(x) = \frac{J}{2}\operatorname{sgn}{(x-\alpha)} + g(x)$$

where $g(x)$ is continuous at $x = \alpha$.

Deduce that Gibbs' Phenomenon occurs at any discontinuity of a piecewise continuously differentiable function.

7. By writing $s_N(x)$ of Question 4 as

$$s_n(x) = \frac{1}{2}\int_0^{x} \frac{\sin 2Nt}{\sin t}\, dt$$

show that

$$\left| s_N\Big(\frac{(n+1)\pi}{2N}\Big) - s_N\Big(\frac{n\pi}{2N}\Big) \right| = \frac{1}{2}\int_0^{\pi/2N} \frac{\sin 2Nt}{\sin(t + n\pi/2N)} dt.$$

Deduce that

$$\max_{x \text{ real}} |s_N(x)| = s_N(\pi/2N)$$

and hence that $s_N(x)$ is uniformly bounded over $N \geq 1$ and x real.

8. Use Question 6 to show that the Fourier series of any piecewise continuously differentiable function has uniformly bounded partial sums over the real line.

Deduce that the series

$$\sum_{2}^{\infty} \frac{\sin nx}{n \log n}$$

is uniformly convergent over all real x.

N.B. This series is not uniformly absolutely convergent over all real x.

Hint Use the Fourier series

$$\sum_{1}^{\infty} \frac{\sin nx}{n} = \frac{\pi - x}{2}$$

valid for $0 < x < 2\pi$.

9. Prove the nth turning point of $s_N(x)$ of Question 4 converges to

$$\frac{1}{2} \int_{0}^{n\pi} \frac{\sin x}{x}\, dx$$

as $N \to \infty$.