Chapter 5
Power Series

5.1 **Notation and Terminology** By a power series we mean a series of the form

$$\sum_0^\infty a_n x^n$$

where x is a variable usually real valued, though in many investigations is better regarded as complex, and a_n is a sequence of constants called **coefficients**. Basic examples are

$$\sum_0^\infty x^n = \frac{1}{1-x}$$

valid for $|x| < 1$, and

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \end{aligned}$$

valid for all real x.

The importance of power series is that they enable arbitrary functions to be approximated by polynomials, and thereby analysed in various ways, in particular computed numerically. Power series also arise as solutions of certain types of ordinary differential equation. The exponential function e^x arises in this way, as do the trigonometric functions $\cos x$, $\sin x$ and more generally hyperbolic functions, Bessel functions and all the other hypergeometric functions. A knowledge of the theory of power series is indispensable to a proper understanding of the behaviour of functions of a complex variable. Some of the terminology relating to power series refers to the complex variable situation. For example the circle of covergence and the radius of convergence. We shall for the most part stay with real variables, though we may occasionally stray into the complex plane to simplify a calculation or illustrate a point.

5.2 **Radius of Convergence** For every power series

$$\sum_0^\infty a_n z^n$$

of a complex variable z there is a **circle of convergence** $|z| = R$ with radius R which has the property that the power series converges for every $|z| < R$ and diverges for every $|z| > R$. This phenomenon manifests itself on the real line as an **interval of convergence** $|x| < R$. We shall still call R the **radius of convergence** for the power series. Whilst it is always true that the series converges for all $|x| < R$ and diverges for all $|x| > R$ there is no general rule about what happens when $|x| = R$, i.e. $x = \pm R$.

For example, for the series

$$\sum_{0}^{\infty} x^n$$

we clearly have $R = 1$ and in this case the series diverges when $x = \pm 1$. On the other hand the series

$$\sum_{0}^{\infty} \frac{x^n}{n^2}$$

also has $R = 1$ but converges when $x = \pm 1$. On the third hand the series

$$\sum_{0}^{\infty} \frac{x^n}{n}$$

again has $R = 1$ but converges when $x = -1$ and diverges when $x = 1$.

To demonstrate formally the existence of the radius of convergence R for a general power series

$$\sum_{0}^{\infty} a_n x^n$$

we start with a definition.

Definition If we have a power series

$$\sum_{0}^{\infty} a_n x^n$$

then we define its **radius of convergence** R to be $R = \sup E$ where E is the set

$$E = \{x : \sum_{0}^{\infty} a_n x^n \text{ converges}\}.$$

Observe that $0 \in E$ so we must have $R \geq 0$. We have $R = 0$ in the case where the power series only converges when $x = 0$. An example of such a series is

$$\sum_{0}^{\infty} n! x^n.$$

It may be that E is unbounded above in which case we say $R = \infty$. An example of a series in this category is the exponential series

$$\sum_{0}^{\infty} \frac{x^n}{n!}$$

which converges for all x.

Lemma If

$$\sum_0^\infty a_n x^n$$

is a power series and E is the set

$$E = \{x : \sum_0^\infty a_n x^n \text{ converges}\}$$

and if $x \in E$ then also $y \in E$ for any $|y| < |x|$.

Proof If

$$\sum_0^\infty a_n x^n$$

converges then the sequence $a_n x^n \to 0$ as $n \to \infty$ so is bounded, say

$$|a_n x^n| \leq M$$

for all $n \geq 1$. It follows that for any $|y| < |x|$ we have

$$|a_n y^n| = |a_n x^n| \left|\frac{y}{x}\right|^n \leq M \left|\frac{y}{x}\right|^n$$

and the series

$$\sum_0^\infty \left|\frac{y}{x}\right|^n$$

converges since it is a geometric series with common ratio

$$\left|\frac{y}{x}\right| < 1.$$

Hence the series

$$\sum_0^\infty a_n y^n$$

converges absolutely by the comparison test. Q.E.D.

Theorem If the power series

$$\sum_0^\infty a_n x^n$$

has radius of convergence R (see above definition) then it converges (absolutely) for all $|x| < R$ and diverges for all $|x| > R$.

Proof For any $|x| < R$ there must exist $y \in E$ such that

$$|x| < y < R.$$

Therefore

$$\sum_0^\infty a_n x^n$$

converges absolutely by the lemma. On the other hand if $|x| > R$ then

$$\sum_0^\infty a_n x^n$$

must diverge since otherwise by the lemma it would have to converge for all y satisfying

$$R < y < |x|$$

which would contradict the definition of R. Q.E.D.

Exercises Find the radius of convergence of the power series

$$\sum_0^\infty a_n x^n$$

where

(i) $a_n =$ the nth Fibonacci number, i.e. $a_0 = 0$, $a_1 = 1$, $a_n = a_{n-1} + a_{n-2}$ $(n \geq 2)$,
(ii) $a_n = 1$ if n prime, $= 0$ if n composite.

5.3 **Uniform Convergence of Power Series** We have established the existence of the **radius of convergence** R of a general power series which has the property that the series converges pointwise on the interval $|x| < R$ but fails to converge pointwise on the intervals $|x| > R$. We now investigate the question of uniform convergence on the interval $|x| < R$.

Theorem If the power series

$$\sum_0^\infty a_n x^n$$

has radius of convergence R then it is almost uniformly convergent on the interval $|x| < R$. By which we mean that if I is the interval $|x| < R$ then the series converges uniformly on any closed subinterval $J \subseteq I$. (See 2.5.)

Proof Choose r satisfying $0 < r < R$ such that $J \subseteq [-r, r]$. Then for all $x \in J$ and all $n \geq 1$ we have

$$|a_n x^n| \leq |a_n| r^n,$$

and the series

$$\sum_{0}^{\infty} |a_n| r^n$$

converges by Theorem 5.2. Hence

$$\sum_{0}^{\infty} a_n x^n$$

converges uniformly on J by the Weierstrass M-Test. (See 4.4.) Q.E.D.

Corollary The sum

$$s(x) = \sum_{0}^{\infty} a_n x^n$$

of a power series with radius of convergence R is continuous on the interval $|x| < R$.

Proof Follows from 2.5 and 4.5. Q.E.D.

Exercises 1. Show that if the power series

$$\sum_{0}^{\infty} a_n x^n$$

has radius of convergence R and if the series converges absolutely at $x = R$ then it is uniformly convergent on the closed interval $|x| \leq R$.

2. Discuss the continuity of the sum of the series

$$\sum_{1}^{\infty} \frac{x^n}{n^2}.$$

5.4 **Differentiation of Power Series** The reader will recall (see 4.7) that to justify differentiating a series term by term one needs to know that the differentiated series is uniformly convergent. If we have a power series

$$\sum_{0}^{\infty} a_n x^n$$

then the differentiated series is

$$\sum_{1}^{\infty} n a_n x^{n-1}$$

which happens to be another power series. It also happens to have the same radius of convergence as the original power series. Before proceeding to a proof of this remarkable fact we prove the following preliminary lemma.

Lemma If the series

$$\sum_0^\infty a_n x^n$$

converges for a particular value of x then the series

$$\sum_1^\infty n a_n y^n$$

converges for any y satisfying $|y| < |x|$.

Proof The sequence $a_n x^n \to 0$ as $n \to \infty$ therefore must be bounded, say

$$|a_n x^n| \le M$$

for all $n \ge 1$. It follows that for any $|y| < |x|$ we must have

$$|n a_n y^n| = |a_n x^n| n \left|\frac{y}{x}\right|^n \le M n \left|\frac{y}{x}\right|^n$$

and the series

$$\sum_1^\infty n \left|\frac{y}{x}\right|^n$$

converges e.g. by the Ratio Test since

$$\left|\frac{y}{x}\right| < 1.$$

Hence the series

$$\sum_1^\infty n a_n y^n$$

converges absolutely by the Comparison Test. Q.E.D.

Corollary If the series

$$\sum_0^\infty a_n x^n$$

has radius of convergence R then the series

$$\sum_1^\infty n a_n x^n$$

also has radius of convergence R.

Proof Suppose the radius of convergence of

$$\sum_{1}^{\infty} na_n x^n$$

is R' and suppose $|x| < R'$. Then

$$\sum_{1}^{\infty} na_n x^n$$

is absolutely convergent and therefore so is

$$\sum_{0}^{\infty} a_n x^n$$

by the Comparison Test. Hence $R \geq R'$.

Now suppose on the other hand that $|x| < R$. Choose y satisfying $|x| < y < R$. Then

$$\sum_{0}^{\infty} a_n y^n$$

converges and therefore by the lemma so does

$$\sum_{1}^{\infty} na_n x^n.$$

Hence $R' \geq R$.

It follows that $R' = R$. Q.E.D.

Theorem If the power series

$$s(x) = \sum_{0}^{\infty} a_n x^n$$

has radius of convergence R then the sum $s(x)$ is differentiable on the interval $|x| < R$ and its derivative is

$$s'(x) = \sum_{1}^{\infty} na_n x^{n-1}.$$

Proof Follows from the fact that the differentiated series is almost uniformly convergent over $|x| < R$. (See 4.7.) Q.E.D.

Exercises 1. Given that

$$\sum_{0}^{\infty} x^n = \frac{1}{1-x}$$

prove that

$$\sum_1^\infty nx^n = \frac{x}{(1-x)^2}$$

for all $|x| < 1$.

2. Find the sum of the series

$$\sum_1^\infty n^2 x^n.$$

5.5 **Taylor Series** Suppose that

$$f(x) = \sum_0^\infty a_n x^n$$

for all $|x| < R$ where R is the radius of convergence of the power series on the right hand side. Then by Theorem 5.4 we have

$$f'(x) = \sum_1^\infty n a_n x^{n-1}$$

for all $|x| < R$. Since the differentiated power series also has radius of convergence R we can apply Theorem 5.4 again to obtain

$$f''(x) = \sum_2^\infty n(n-1) a_n x^{n-2}$$

for all $|x| < R$. In fact, $f(x)$ must have derivatives of all orders over the interval $|x| < R$, the kth derivative being given by

$$f^{(k)}(x) = \sum_k^\infty n(n-1)\dots(n-k+1) a_n x^{n-k}.$$

If we put $x = 0$ in this formula we get

$$a_k = \frac{1}{k!} f^{(k)}(0).$$

Now suppose instead that we start with a function $f(x)$ known to have derivatives of all orders on the interval $|x| < R$ for some $R > 0$. Then we define the **Maclaurin coefficients** of $f(x)$ to be

$$a_n = \frac{1}{n!} f^{(n)}(0)$$

and the **Maclaurin series** of $f(x)$ to be

$$\sum_0^\infty a_n x^n$$

where a_n are the Maclaurin coefficients of $f(x)$. We then ask whether

$$f(x) = \sum_0^\infty a_n x^n$$

for $|x| < R$. Whilst in a number of cases the answer to this question is yes, it turns out that in the general case the answer is no. Even when the answer is no, however, the partial sums of the Maclaurin series still provide a useful approximation to the function $f(x)$ at least for small x.

For $f(x) = e^x$, $\sin x$, $\cos x$ the Maclaurin series are the familiar series listed in Section 5.1 and they do converge to their respective generating functions. Consider however $f(x) = \tan^{-1} x$. The Maclaurin series is

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

(see 5.6) which only converges if $|x| \leq 1$, whereas the function $\tan^{-1} x$ has derivatives of all orders over the whole real line. Even when the Maclaurin series converges it may not converge to its generating function. For example, the function

$$\begin{aligned} f(x) &= e^{-1/x^2} \quad && (x \neq 0) \\ &= 0 && (x = 0) \end{aligned}$$

has $f^{(n)}(0) = 0$ for all $n \geq 0$ so the Maclaurin series of $f(x)$ vanishes identically and therefore converges trivially but its sum only $= f(x)$ at $x = 0$.

Maclaurin series are actually a special case of Taylor series. Given a function $f(x)$ known to have derivatives of all orders at $x = c$ we define its **Taylor coefficients** at c to be

$$a_n = \frac{1}{n!} f^{(n)}(c)$$

and its **Taylor series** at c to be

$$\sum_0^\infty a_n (x - c)^n$$

where a_n are the Taylor coefficients at c. The story for Taylor series is similar to that for Maclaurin series.

Exercises 1. Find the Taylor series of $\log x$ at $x = 1$.
2. Find the Taylor series of $\cos x$ at $x = \pi/2$.

5.6 **Integration of Power series** The problem of justifying integration of an infinite series is usually considerably simpler than that of differentiating it. And power series are no exception. The formal theorem is as follows.

Theorem If the power series

$$f(x) = \sum_0^\infty a_n x^n$$

has radius of convergence R then for any $|x| < R$ we have

$$\int_0^x f(t)dt = \sum_0^\infty a_n \frac{x^{n+1}}{n+1}.$$

Proof The result follows from 5.2 and 4.7. Q.E.D.

Application The expansion

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

is valid for all $|x| < 1$.

Proof The geometric series

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

has radius of convergence $R = 1$, so integrating term by term we have

$$\log(1+x) = \int_0^x \frac{dt}{1+t} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

for all $|x| < 1$. Q.E.D.

Exercise Prove similarly that the expansion

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

is valid for all $|x| < 1$.

Hint Integrate the geometric series

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

term by term.

5.7 **Abel's Theorem** It is an easy matter to show that if the power series

$$\sum_0^\infty a_n x^n$$

has radius of convergence R and if the series

$$\sum_0^\infty a_n R^n$$

is absolutely convergent then the power series converges uniformly absolutely over the closed interval $|x| \leq R$. (Use the Weierstrass M-Test.) We now show that a similar result is true with the word 'absolutely' deleted at every occurrence.

Theorem If the power series

$$\sum_0^\infty a_n x^n$$

has radius of convergence R and the series

$$\sum_0^\infty a_n R^n$$

converges then the power series converges uniformly over the interval $0 \leq x \leq R$.

Proof We use Abel's Test for uniform convergence. (See 4.6.) We write

$$\sum_0^\infty a_n x^n = \sum_0^\infty a_n R^n \left(\frac{x}{R}\right)^n$$

and observe that the series

$$\sum_0^\infty a_n R^n$$

converges uniformly trivially since it is independent of x, and the sequence $(x/R)^n$ decreases (as n increases) and is uniformly bounded over $0 \leq x \leq R$ since

$$0 \leq \left(\frac{x}{R}\right)^n \leq 1$$

for all $n \geq 0$ and all $0 \leq x \leq R$. Q.E.D.

Observe This theorem cannot be proved with Dirichlet's Test since the sequence $(x/R)^n$ doesn't $\to 0$ uniformly over $0 \leq x \leq R$.

Corollary Abel's Theorem If the series

$$\sum_0^\infty a_n$$

converges then the power series

$$\sum_0^\infty a_n x^n$$

is a continuous function of x over the interval $0 \leq x \leq 1$. In particular

$$\sum_0^\infty a_n x^n \to \sum_0^\infty a_n$$

as $x \to 1$ from below.

Application The expansion

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

is valid for $-1 < x \leq 1$, in particular

$$1 - \frac{1}{2} + \frac{1}{3} - \dots = \log 2.$$

Proof After 5.6 we only have to prove validity at $x = 1$, and this follows from Abel's Theorem as the series

$$1 - \frac{1}{2} + \frac{1}{3} - \dots$$

converges by Dirichlet's Test. (See 3.4.) Q.E.D.

Exercise Prove similarly that the expansion

$$\tan^{-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

is valid for $|x| \leq 1$.

Deduce that (Gregory's Series)

$$1 - \frac{1}{3} + \frac{1}{5} - \dots = \frac{\pi}{4}.$$

5.8 **Multiplication of Series** An important theorem in elementary analysis states that if the series

$$\sum_0^\infty a_n, \quad \sum_0^\infty b_n$$

both converge absolutely then their **Cauchy Product**

$$\sum_0^\infty c_n$$

where

$$c_n = \sum_0^n a_r b_{n-r}$$

also converges absolutely and its sum

$$\sum_0^\infty c_n = \sum_0^\infty a_n \sum_0^\infty b_n$$

is equal to the product of the sums of the original series.

This result is not true if one assumes that the series

$$\sum_0^\infty a_n, \quad \sum_0^\infty b_n$$

merely converge. The standard counter-example is

$$\sum_0^\infty a_n = \sum_0^\infty b_n = \sum_0^\infty \frac{(-1)^n}{\sqrt{n+1}}.$$

Here we have

$$\begin{aligned} |c_n| &= \left| \sum_0^n a_r b_{n-r} \right| \\ &= \sum_0^n \frac{1}{\sqrt{r+1}\sqrt{n-r+1}} \\ &\geq \sum_0^n \frac{2}{(r+1)+(n-r+1)}, \end{aligned}$$

using the G.M $\leq$ A.M. inequality,

$$\begin{aligned} &= \sum_0^n \frac{2}{n+2} \\ &= 2\,\frac{n+1}{n+2} \\ &\to 2 \end{aligned}$$

as $n \to \infty$. So the Cauchy Product

$$\sum_0^\infty c_n$$

diverges in this case.

Nevertheless, there is a theorem which can be proved in the absence of absolute convergence. It depends on Abel's Theorem (see 5.7) and goes as follows.

Theorem If the three series

$$\sum_0^\infty a_n, \quad \sum_0^\infty b_n, \quad \sum_0^\infty c_n$$

all converge, where

$$c_n = \sum_0^n a_r b_{n-r},$$

then

$$\sum_0^\infty c_n = \sum_0^\infty a_n \sum_0^\infty b_n.$$

Proof We certainly have

$$\sum_0^\infty c_n x^n = \sum_0^\infty a_n x^n \sum_0^\infty b_n x^n$$

for all $|x| < 1$ since the series on the right hand side are both absolutely convergent. So the result follows by letting $x \to 1$ from below and appealing to Abel's Theorem. Q.E.D.

Observe That in the counter-example we gave above the Cauchy Product **diverges**.

Exercise Find the value of

$$\lim_{n\to\infty} \sum_0^n \frac{1}{\sqrt{r+1}\sqrt{n-r+1}}.$$

Hint Approximate to a suitable definite integral.

5.9 **Calculation of** π We have already observed that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

(see 5.7) but this series is not a lot of use for calculating π as it converges so slowly. Faster converging series have been constructed by means of the formula

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$$

and the Maclaurin series

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

valid for $|x| \le 1$. The most celebrated of these is Machin's Formula

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

which offers a good compromise between the demands of accuracy and simplicity.

Exercises Verify the following formulae.

$$\begin{aligned}\frac{\pi}{4} &= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}\\ &= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}\end{aligned}$$

5.10 **The Binomial Theorem** We conclude this chapter with an account of the Binomial Theorem. We shall consider the theorem in the form

$$(1+x)^\alpha = \sum_0^\infty \binom{\alpha}{n} x^n$$

where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}.$$

For $\alpha =$ a positive integer the Binomial Theorem is a theorem of algebra and is proved by induction. For other α the Binomial Theorem is a theorem of analysis and its proof requires considerable subterfuge.

The first thing to observe is that the series

$$\sum_0^\infty \binom{\alpha}{n} x^n$$

is the Maclaurin series of $(1+x)^\alpha$. This fact in itself is of course no guarantee either that the series converges or that even if it does converge its sum should be $(1+x)^\alpha$.

We can however easily compute the radius of convergence of the series

$$\sum_0^\infty \binom{\alpha}{n} x^n$$

by means of the Ratio Test. In fact we have

$$\left| \binom{\alpha}{n+1} x^{n+1} \Big/ \binom{\alpha}{n} x^n \right| = \left| \frac{\alpha - n}{n+1}\, x \right| \to |x|$$

as $n \to \infty$ which tells us immediately that the radius of convergence is 1.

We shall divide the Binomial Theorem into three cases. For the case $|x| < 1$ we shall use the fact that the power series can be differentiated term by term to show that its sum satisfies a certain differential equation which also happens to be satisfied by the function $(1+x)^\alpha$. For the cases $x = \pm 1$ we use Abel's Theorem.

Theorem (Case $|x| < 1$) For all real α we have

$$(1+x)^\alpha = \sum_0^\infty \binom{\alpha}{n} x^n$$

for all $|x| < 1$.

Proof Write

$$f(x) = \sum_0^\infty \binom{\alpha}{n} x^n$$

for all $|x| < 1$. Then by Theorem 5.4 we have for all $|x| < 1$

$$\begin{aligned} f'(x) &= \sum_1^\infty n \binom{\alpha}{n} x^{n-1} \\ &= \sum_1^\infty \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{(n-1)!} x^{n-1} \\ &= \alpha \sum_1^\infty \binom{\alpha-1}{n-1} x^{n-1} \\ &= \alpha \sum_0^\infty \binom{\alpha-1}{n} x^n. \end{aligned}$$

Therefore

$$\begin{aligned} (1+x)f'(x) &= \alpha \sum_0^\infty \binom{\alpha-1}{n} x^n + \alpha \sum_1^\infty \binom{\alpha-1}{n-1} x^n \\ &= \alpha \left(1 + \sum_1^\infty \binom{\alpha}{n} x^n \right) \\ &= \alpha f(x) \end{aligned}$$

using the identity

$$\binom{\alpha-1}{n-1}+\binom{\alpha-1}{n}=\binom{\alpha}{n}.$$

It follows that $f(x)$ satisfies the differential equation

$$(1+x)f'(x)=\alpha f(x)$$

for all $|x|<1$, together with the initial condition $f(0)=1$. The differential equation can be written in the form

$$\frac{d}{dx}(1+x)^{-\alpha}f(x)=0$$

which gives

$$f(x)=A(1+x)^{\alpha}$$

where A is a constant which $=1$ since $f(0)=1$. Hence we have

$$f(x)=(1+x)^{\alpha}$$

for all $|x|<1$ as required. Q.E.D.

It remains to consider what happens at the end points $x=\pm 1$ of the interval of convergence of the series

$$\sum_0^{\infty}\binom{\alpha}{n}x^n.$$

By Abel's Theorem (see 5.7) this is simply a question of convergence of the series

$$\sum_0^{\infty}\binom{\alpha}{n},\quad \sum_0^{\infty}(-1)^n\binom{\alpha}{n}$$

except that this is not such a simple question! We shall prove the following two theorems.

Theorem (Case $x=1$) For all $\alpha>-1$ we have

$$\sum_0^{\infty}\binom{\alpha}{n}=2^{\alpha}.$$

Theorem (Case $x=-1$) For all $\alpha>0$ we have

$$\sum_0^{\infty}(-1)^n\binom{\alpha}{n}=0.$$

Both theorems follow easily from the following Lemma.

Lemma For any real α there exist positive constants A, B such that

$$\frac{A}{n^{\alpha+1}} \leq \left|\binom{\alpha}{n}\right| \leq \frac{B}{n^{\alpha+1}}$$

for all $n \geq 0$.

N.B. The binomial coefficient

$$\binom{\alpha}{n}$$

alternates in sign for large n so

$$\sum_0^\infty \binom{\alpha}{n}$$

is ultimately an alternating series whilst

$$\sum_0^\infty (-1)^n \binom{\alpha}{n}$$

has terms ultimately of constant sign.

To prove the Lemma we shall require two inequalities.

Inequality 1 For all $|x| \leq 1/2$ we have

$$|\log(1+x) - x| \leq x^2.$$

Inequality 2 For all $N \geq 1$ we have

$$\left|\sum_1^N \frac{1}{n} - \log N\right| \leq 1.$$

Proof of Inequality 1 From 5.6 we have for all $|x| \leq 1/2$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

and therefore

$$\begin{aligned} |\log(1+x) - x| &= \left|-\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right| \\ &\leq x^2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right) \\ &= x^2. \end{aligned}$$

Proof of Inequality 2 Observe that for all $n \geq 1$

$$\frac{1}{n+1} \leq \int_n^{n+1} \frac{dx}{x} \leq \frac{1}{n}$$

and therefore

$$\sum_1^{N-1} \frac{1}{n+1} \leq \int_1^N \frac{dx}{x} \leq \sum_1^{N-1} \frac{1}{n},$$

$$\sum_2^{N} \frac{1}{n} \leq \log N \leq \sum_1^{N-1} \frac{1}{n},$$

$$\frac{1}{N} \leq \sum_1^{N} \frac{1}{n} - \log N \leq 1.$$

Proof of the Lemma Observe that

$$\begin{aligned}\binom{\alpha}{N} &= \frac{\alpha(\alpha-1)\dots(\alpha-N+1)}{N!} \\ &= \prod_1^N \left(\frac{\alpha+1}{n} - 1\right)\end{aligned}$$

and therefore

$$\log\left|\binom{\alpha}{N}\right| = \sum_1^N \log\left|\frac{\alpha+1}{n} - 1\right|.$$

By Inequality 1 we have for $n > 2|\alpha+1|$

$$\left|\log\left|\frac{\alpha+1}{n} - 1\right| + \frac{\alpha+1}{n}\right| \leq \left(\frac{\alpha+1}{n}\right)^2$$

and therefore

$$\left|\sum_1^N \log\left|\frac{\alpha+1}{n} - 1\right| + (\alpha+1)\sum_1^N \frac{1}{n}\right| \leq \sum_1^N \left(\frac{\alpha+1}{n}\right)^2 + A$$

where A is a constant. It follows that

$$\left|\log\left|\binom{\alpha}{N}\right| + (\alpha+1)\log N\right| \leq B$$

where B is another constant by Inequality 2 and the fact that the series

$$\sum_1^\infty \frac{1}{n^2}$$

converges. The Lemma now follows (with different A, B) by exponentiating. Q.E.D.

Proof of the Theorem (Case $x = 1$) Clearly

$$\left|\binom{\alpha}{n}\right|$$

is ultimately decreasing, and it $\to 0$ as $n \to \infty$ for $\alpha > -1$ by the Lemma. Hence the series

$$\sum_0^\infty \binom{\alpha}{n}$$

converges by Dirichlet's Test. (See 3.4.) Its sum must be 2^α by Abel's Theorem. (See 5.7.) Q.E.D.

Proof of the Theorem (Case $x = -1$) The series

$$\sum_0^\infty (-1)^n \binom{\alpha}{n}$$

has terms ultimately of constant sign and so converges for $\alpha > 0$ by the Lemma and the Comparison Test. Its sum must be zero by Abel's Theorem. (See 5.7.) Q.E.D.

N.B. The Lemma also shows that the series

$$\sum_0^\infty \binom{\alpha}{n}$$

diverges for all $\alpha \leq -1$. The series

$$\sum_0^\infty (-1)^n \binom{\alpha}{n}$$

cannot converge for $\alpha < 0$ since if it did then Abel's Theorem would imply its sum to be

$$\lim_{x \to -1} (1 + x)^\alpha = \infty$$

which is absurd.

5.11 **Miscellaneous Exercises** 1. Prove that if

$$\sum_0^\infty a_n x^n$$

is uniformly convergent on the interval $|x| < 1$ then

$$\sum_0^\infty a_n$$

must converge.

2. Show that if

$$\sum_0^\infty a_n x^n$$

has a finite limit as $x \to 1$ from below then it does not necessarily follow that

$$\sum_0^\infty a_n$$

converges.

3. Prove the following integrals have the values stated.

$$\int_0^1 \frac{\log(1-x)}{x} dx = -\sum_1^\infty \frac{1}{n^2} \qquad \int_0^1 \frac{\log(1+x)}{x} dx = \sum_1^\infty \frac{(-1)^n}{n^2}$$

4. Prove that

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \dots$$

for all $|x| < 1$ by integrating the binomial expansion of

$$\frac{1}{\sqrt{1-x^2}}$$

term by term.

Is this expansion valid for $x = \pm 1$?