Chapter 4
Uniform Convergence of Series

4.1 **Preamble** We say the series of functions

$$\sum_1^\infty f_n(x)$$

converges **pointwise** on an interval I if the sequence

$$s_N(x) = \sum_1^N f_n(x)$$

of **partial sum functions** converges pointwise on I. Which is to say that there exists a **sum function** $s(x)$ such that

$$s_N(x) \to s(x)$$

pointwise on I. We say

$$\sum_1^\infty f_n(x)$$

is **uniformly** convergent on I if

$$s_N(x) \to s(x)$$

uniformly on I.

For example, suppose $f_n(x) = x^n$. Then

$$\begin{aligned} s_N(x) = \sum_1^N x^n &= \frac{x - x^{N+1}}{1-x} \quad (x \neq 1) \\ &= N \quad (x = 1) \end{aligned}$$

which converges to

$$s(x) = \frac{x}{1-x}$$

pointwise over the interval $I = (-1, 1)$. However,

$$\begin{aligned} M_N &= \sup_{|x|<1} |s_N(x) - s(x)| \\ &= \sup_{|x|<1} \left| \frac{x^{N+1}}{1-x} \right| \\ &= \infty \end{aligned}$$

for all N so $s_N(x) \not\to s(x)$ uniformly on I. Therefore the series

$$\sum_1^\infty x^n$$

is pointwise convergent but not uniformly convergent on $|x| < 1$.

Theorems analogous to those of Chapter 2 also hold for uniformly convergent series. For example, if all the terms of a uniformly convergent series are continuous functions then so is the sum function. Also it is legitimate to integrate the series term by term, and to differentiate it term by term provided the terms are all differentiable and the differentiated series is uniformly convergent. See 4.7 for details.

We shall be mainly concerned in this Chapter with tests for uniform convergence of series. We shall give applications to Power Series and to Trigonometric Series in Chapters 5 and 6.

The most important test we shall consider is the Weierstrass M-Test. This test has the merit of reducing the question of whether a series of functions converges uniformly to that of whether a series of positive constants converges. And of course elementary analysis provides us with a large number of tests for resolving this problem. The Weierstrass M-Test is a test for uniform **absolute** convergence, meaning

$$\sum_{1}^{\infty} |f_n(x)|$$

converges uniformly. There are analogues of Dirichlet's and Abel's tests available for testing for uniform **conditional** convergence. We also consider tests for non-uniform convergence. These include the Uniform Non-Null Test and the Uniform Cauchy Criterion. Also non-continuity of the sum function in the case where the terms are continuous.

The logical order of presentation requires us to cover the Uniform Cauchy Criterion first since the proof of the Weierstrass M-Test depends on it. We would not wish to discourage the reader from taking the tests in a different order if he or she prefers.

Exercises Is the series

$$\sum_{1}^{\infty} x^n$$

uniformly convergent over (i) $-1 < x \le 0$? (ii) $0 \le x \le 1/2$?

4.2 **Uniform Cauchy Criterion** In accordance with the tradition laid down in Chapter 3 we shall prove the Uniform Cauchy Criterion in the context of sequences first and then proceed to series.

Theorem The sequence of functions $f_n(x)$ is uniformly convergent on the interval I if and only if given $\epsilon > 0$ there exists N such that

$$|f_m(x) - f_n(x)| < \epsilon$$

for all $m, n > N$ and all $x \in I$.

Proof Suppose $f_n(x) \to f(x)$ uniformly on I. Then given $\epsilon > 0$ we can choose N such that

$$|f_m(x) - f(x)| < \epsilon/2$$

for all $n > N$ and all $x \in I$. Therefore for any $m, n > N$ and $x \in I$ we have

$$\begin{aligned} |f_m(x) - f_n(x)| &\leq |f_m(x) - f(x)| + |f(x) - f_n(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

On the other hand suppose $f_n(x)$ satisfies the Uniform Cauchy Criterion. Then for each fixed $x \in I$ the numerical sequence $f_n(x)$ satisfies the Ordinary Cauchy Criterion so there exists a limit which we can call $f(x)$. In other words there exists a function $f(x)$ defined on I such that $f_n(x) \to f(x)$ pointwise on I. We show $f_n(x) \to f(x)$ uniformly on I. In fact, for any given $\epsilon > 0$ we can choose N such that

$$|f_m(x) - f_n(x)| < \epsilon$$

for all $m, n > N$ and all $x \in I$. Fix $n > N$ and $x \in I$ and let $m \to \infty$. We get

$$|f(x) - f_n(x)| \leq \epsilon$$

for all $n > N$ and all $x \in I$. Q.E.D.

Corollary The series

$$\sum_1^\infty f_n(x)$$

is uniformly convergent on the interval I if and only if given $\epsilon > 0$ there exists N such that

$$\left| \sum_P^Q f_n(x) \right| < \epsilon$$

for all $P, Q > N$ and all $x \in I$.

Proof Apply the theorem to the sequence

$$s_N(x) = \sum_1^N f_n(x)$$

of partial sums. Q.E.D.

4.3 **Uniform Absolute Convergence** We say

$$\sum_1^\infty f_n(x)$$

is uniformly **absolutely** convergent on the interval I if

$$\sum_1^\infty |f_n(x)|$$

is uniformly convergent on I.

Theorem Uniform absolute convergence implies uniform convergence.

Proof Suppose

$$\sum_1^\infty |f_n(x)|$$

is uniformly convergent on I and that $\epsilon > 0$ is given. Then there exists N such that

$$\sum_P^Q |f_n(x)| < \epsilon$$

for all $P, Q > N$ and all $x \in I$. But therefore also

$$\left| \sum_P^Q f_n(x) \right| \leq \sum_P^Q |f_n(x)| < \epsilon$$

for all $P, Q > N$ and all $x \in I$. Hence

$$\sum_1^\infty f_n(x)$$

is uniformly convergent on I. Q.E.D.

4.4 **The Weierstrass M-Test** This is easily the most useful of the tests for uniform convergence and is the easiest to apply. We present it in the form of a theorem.

Theorem The series

$$\sum_1^\infty f_n(x)$$

is uniformly absolutely convergent over the interval I if the series

$$\sum_1^\infty M_n$$

converges where

$$M_n = \sup_{x \in I} |f_n(x)|.$$

Proof Suppose $\epsilon > 0$ is given. Then by the ordinary Cauchy Criterion there exists N such that

$$\sum_P^Q M_n < \epsilon$$

for all $P, Q > N$. Therefore also

$$\sum_P^Q |f_n(x)| \leq \sum_P^Q M_n < \epsilon$$

for all $P, Q > N$ and all $x \in I$. Hence

$$\sum_1^\infty f_n(x)$$

is uniformly absolutely convergent over I by the Uniform Cauchy Criterion. Q.E.D.

Example The series

$$\sum_1^\infty x^n$$

is almost uniformly absolutely convergent over $|x| < 1$.

Proof If I is the interval $|x| \leq 1 - \epsilon$ then

$$M_n = \sup_{x \in I} |x^n| = (1 - \epsilon)^n$$

and

$$\sum_1^\infty (1 - \epsilon)^n$$

converges. Therefore by the Weierstrass M-Test the series

$$\sum_1^\infty x^n$$

converges uniformly absolutely on I. Q.E.D.

Exercises Use the Weierstrass M-Test to show the following series are uniformly absolutely convergent over $0 \leq x \leq 1$.

$$\sum_1^\infty \frac{x^n}{n^2} \qquad \sum_1^\infty x^2 e^{-nx}$$

4.5 **Tests for Non-Uniform Convergence** Consider the example $f_n(x) = xe^{-nx}$. The series

$$\sum_1^\infty f_n(x) = \sum_1^\infty xe^{-nx}$$

is pointwise convergent over $x \geq 0$. Applying the Weierstrass M-Test we have

$$M_n = \sup_{x \geq 0} xe^{-nx} = 1/ne$$

attained at $x = 1/n$. But the series

$$\sum_1^\infty M_n = \sum_1^\infty \frac{1}{ne}$$

diverges. Can we deduce that the series

$$\sum_1^\infty xe^{-nx}$$

is not uniformly convergent over $x \geq 0$?

Unfortunately we can't. The Weierstrass M-Test only tells us what happens if the series

$$\sum_1^\infty M_n$$

converges. It says nothing about what happens when

$$\sum_1^\infty M_n$$

diverges. To settle the question of the uniform convergence of

$$\sum_1^\infty xe^{-nx}$$

over $x \geq 0$ we have to turn to other tests.

Option 1 We can use the Uniform Cauchy Criterion. To show non-uniform convergence of

$$\sum_1^\infty f_n(x)$$

over the interval I we have to show the Cauchy Criterion fails. This is equivalent to showing that

$$\sum_P^Q f_n(x)$$

fails to $\to 0$ uniformly over I as P, Q both $\to \infty$. It is sufficient to show

$$\sum_{N+1}^{2N} f_n(x) \not\to 0$$

uniformly over I as $N \to \infty$.

In the case $f_n(x) = xe^{-nx}$ over $x \geq 0$ we have

$$\begin{aligned} M_N &= \sup_{x\geq 0} \sum_{N+1}^{2N} xe^{-nx} \\ &\geq \text{value at } x = 1/N \\ &= \sum_{N+1}^{2N} \frac{1}{N} e^{-n/N} \\ &\geq \sum_{N+1}^{2N} \frac{1}{N} e^{-2N/N} \\ &= 1/e^2 \\ &\not\to 0 \end{aligned}$$

as $N \to \infty$. Therefore

$$\sum_{N+1}^{2N} xe^{-nx} \not\to 0$$

uniformly over $x \geq 0$ and hence

$$\sum_{1}^{\infty} xe^{-nx}$$

is not uniformly convergent over $x \geq 0$.

Exercises Use the above method to show non-uniform convergence of the following series over $0 \leq x \leq 1$.

$$\sum_{1}^{\infty} xe^{-nx^2} \qquad \sum_{1}^{\infty} x^n(1-x)$$

Option 2 is of limited application, though very effective when it can be applied. It is actually a special case of Option 1, but we shall present it as a theorem in its own right.

Theorem If the series

$$\sum_{1}^{\infty} f_n(x)$$

is uniformly convergent over the interval I then its nth term $f_n(x) \to 0$ uniformly over I.

Proof One can either observe that

$$f_N(x) = \sum_N^N f_n(x)$$

and use the Cauchy Criterion, or argue directly as follows. Suppose

$$s_N(x) = \sum_1^N f_n(x) \to s(x)$$

uniformly over I. Then

$$\begin{aligned} f_n(x) &= s_n(x) - s_{n-1}(x) \quad (n \geq 2) \\ &\to 0 - 0 \\ &= 0 \end{aligned}$$

uniformly over I. Q.E.D.

To show non-uniform convergence of a series

$$\sum_1^\infty f_n(x)$$

over an interval I one simply has to show $f_n(x) \not\to 0$ uniformly over I. For example,

$$\sum_1^\infty x^n$$

can be shown to be non-uniformly convergent over $|x| < 1$ by observing that $x^n \not\to 0$ uniformly over $|x| < 1$.

Exercises Show the following series are not uniformly convergent over $x \geq 0$ by the above method.

$$\sum_1^\infty e^{-nx} \sin nx \qquad \sum_1^\infty \left(\frac{x}{x+1}\right)^n$$

Option 3 is also of limited application but for a different reason. It depends on the following theorem.

Theorem If $f_n(x)$ is continuous on the interval I for all n and if

$$\sum_1^\infty f_n(x)$$

converges uniformly on I then the sum

$$s(x) = \sum_1^\infty f_n(x)$$

is also continuous on I.

Proof The Nth partial sum

$$s_N(x) = \sum_1^N f_n(x)$$

is continuous on I from elementary analysis. Hence the result follows from 2.2. Q.E.D.

It follows from this theorem that if the terms of a pointwise convergent series are all continuous but the sum isn't continuous then the convergence cannot be uniform. Unfortunately it is only on very rare occasions that it is possible to obtain a closed form for the sum of a series.

One example where there is a closed form available for the sum of the series is

$$\begin{aligned} \sum_1^\infty xe^{-nx} &= \frac{x}{e^x - 1} \quad (x > 0), \\ &= 0 \quad (x = 0). \end{aligned}$$

Observe that by l'Hôpital's rule

$$\lim_{x \to 0} \frac{x}{e^x - 1} = \lim_{x \to 0} \frac{1}{e^x} = 1$$

so the sum is discontinuous at $x = 0$. Hence we have another proof of the non-uniform convergence of

$$\sum_1^\infty xe^{-nx}$$

over $x \geq 0$.

Exercises Sum the following series and comment on their uniform convergence over $x \geq 0$.

$$\sum_1^\infty xe^{-nx^2} \qquad \sum_1^\infty e^{-nx} \sin nx$$

4.6 **Tests for Uniform Conditional Convergence** In this section we prove the uniform analogues of Dirichlet and Abel's tests. (See 3.4.)

Uniform Dirichlet Test The series

$$\sum_{1}^{\infty} a_n(x)b_n(x)$$

converges uniformly over the interval I if

(i) there exists M such that

$$\left|\sum_{1}^{N} a_n(x)\right| \le M$$

for all $N \ge 1$ and all $x \in I$,

(ii) $b_n(x) \ge b_{n+1}(x)$ for all $n \ge 1$ and all $x \in I$,

(iii) $b_n(x) \to 0$ as $n \to \infty$ uniformly over I.

Proof Write

$$s_N(x) = \sum_{1}^{N} a_n(x).$$

Then for any $P, Q \ge 1$ and any $x \in I$ we have

$$\begin{aligned}\sum_{P}^{Q} a_n(x)b_n(x) &= \sum_{P}^{Q} \big(s_n(x) - s_{n-1}(x)\big)b_n(x) \\ &= -s_{P-1}(x)b_P(x) + \sum_{P}^{Q-1} s_n(x)\big(b_n(x) - b_{n+1}(x)\big) + s_Q(x)b_Q(x)\end{aligned}$$

and therefore

$$\begin{aligned}\left|\sum_{P}^{Q} a_n(x)b_n(x)\right| &\le Mb_P(x) + M\sum_{P}^{Q-1} \big(b_n(x) - b_{n+1}(x)\big) + Mb_Q(x) \\ &= 2Mb_P(x).\end{aligned}$$

Now suppose $\epsilon > 0$ is given. Then we can choose N such that

$$|b_n(x)| \le \epsilon/2M$$

for all $n > N$ and all $x \in I$. Therefore

$$\left|\sum_{P}^{Q} a_n(x)b_n(x)\right| < \epsilon$$

for all $P, Q > N$ and all $x \in I$. Hence

$$\sum_{1}^{\infty} a_n(x)b_n(x)$$

converges uniformly over I by the Uniform Cauchy Criterion. Q.E.D.

As an example of the application of the Uniform Dirichlet Test we show

$$\sum_1^\infty \frac{\sin nx}{n}$$

is almost uniformly convergent over $0 < x < 2\pi$. We take $a_n(x) = \sin nx$ and $b_n(x) = 1/n$. If I is the interval $\epsilon \leq x \leq 2\pi - \epsilon$ then

$$\left|\sum_1^N \sin nx\right| \leq \frac{1}{\sin x/2} \leq \frac{1}{\sin \epsilon/2}$$

for all $N \geq 1$ and all $x \in I$. (See 3.4.) Clearly $b_n(x) \geq b_{n+1}(x)$ for all $n \geq 1$ and all $x \in I$. Also $b_n(x) \to 0$ uniformly over I trivially since $b_n(x)$ is independent of x. So the conditions of Dirichlet's Test for Uniform Convergence are fulfilled and hence

$$\sum_1^\infty \frac{\sin nx}{n}$$

converges uniformly over $\epsilon \leq x \leq 2\pi - \epsilon$.

Abel's Test for Uniform Convergence The series

$$\sum_1^\infty a_n(x) b_n(x)$$

converges uniformly over the interval I if

(i) the series

$$\sum_1^\infty a_n(x)$$

converges uniformly over I,

(ii) $b_n(x) \geq b_{n+1}(x)$ for all $n \geq 1$ and all $x \in I$,

(iii) there exists M such that

$$|b_n(x)| \leq M$$

for all $n \geq 1$ and all $x \in I$.

Proof We cannot use Dirichlet's test (c.f. 3.4) since whilst

$$\lim_{n\to\infty} b_n(x)$$

certainly exists pointwise over I there is no guarantee that $b_n(x) \to b(x)$ uniformly over I.

Suppose $\epsilon > 0$ is given. Then we can choose N such that

$$\left| \sum_P^Q a_n(x) \right| < \epsilon / M$$

for all $P, Q > N$ and all $x \in I$. If we write, for $N \geq P$,

$$\sigma_N(x) = \sum_P^N a_n(x)$$

then we have

$$\begin{aligned} \sum_P^Q a_n(x) b_n(x) &= \sigma_P(x) b_P(x) + \sum_{P+1}^Q \big(\sigma_n(x) - \sigma_{n-1}(x)\big) b_n(x) \\ &= \sum_P^{Q-1} \sigma_n(x) \big(b_n(x) - b_{n+1}(x)\big) + \sigma_Q(x) b_Q(x) \end{aligned} .$$

Therefore

$$\begin{aligned} \left| \sum_P^Q a_n(x) b_n(x) \right| &\leq \frac{\epsilon}{M} \sum_P^{Q-1} \big(b_n(x) - b_{n+1}(x)\big) + \frac{\epsilon}{M} b_Q(x) \\ &= \frac{\epsilon}{M} b_P(x) \\ &\leq \epsilon \end{aligned}$$

for all $P, Q > N$ and all $x \in I$. Hence

$$\sum_1^\infty a_n(x) b_n(x)$$

converges uniformly over I. Q.E.D.

As an application of Abel's Test for Uniform Convergence we show

$$\sum_1^\infty (-1)^n \frac{x^n}{n}$$

converges uniformly over $0 \leq x \leq 1$. In fact, we take $a_n(x) = (-1)^n / n$ and $b_n(x) = x^n$. The series

$$\sum_1^\infty \frac{(-1)^n}{n}$$

converges therefore trivially uniformly over any interval since the nth term is independent of x. Also $b_n(x) \geq b_{n+1}(x)$ and $0 \leq b_n(x) \leq 1$ for all $n \geq 1$ and all $0 \leq x \leq 1$. So the conditions of Abel's Test are satisfied.

Exercises 1. Show

$$\sum_{1}^{\infty}(-1)^n\frac{x^n}{n}$$

is uniformly convergent over $0 \le x \le 1$ by taking $a_n(x) = (-1)^n x^n$ and $b_n(x) = 1/n$ and using Dirichlet's Test for Uniform Convergence.

2. Discuss the uniform convergence of the series

$$\sum_{1}^{\infty}\frac{x^n \sin nx}{n}$$

over $0 \le x \le 1$.

4.7 **Analytic Consequences of Uniform Convergence** We proved in 4.5 that the sum of a uniformly convergent series of continuous functions is continuous. It now remains to consider the consequences of uniform convergence for differentiation and integration of series.

Differentiation Term by Term If $f_n(x)$ is a sequence of differentiable functions on an interval I and if

(i) the series

$$\sum_{1}^{\infty} f_n(x)$$

is pointwise convergent on I,

(ii) the series

$$\sum_{1}^{\infty} f_n'(x)$$

is uniformly convergent on I, then

$$s(x) = \sum_{1}^{\infty} f_n(x)$$

is differentiable on I and its derivative is

$$s'(x) = \sum_{1}^{\infty} f_n'(x).$$

Proof If we write

$$s_N(x) = \sum_{1}^{N} f_n(x)$$

then $s_N(x)$ is differentiable on I and

$$s'_N(x) = \sum_1^N f'_n(x).$$

Also $s_N(x) \to s(x)$ pointwise on I and the sequence $s'_N(x)$ is uniformly convergent on I. Hence the result follows from 2.4. Q.E.D.

Integration Term by Term If $f_n(x)$ is a sequence of continuous functions on the interval $[a, b]$ such that the series

$$\sum_1^\infty f_n(x)$$

converges uniformly on $[a, b]$ then the sum

$$s(x) = \sum_1^\infty f_n(x)$$

is continuous on I and

$$\int_a^b s(x)dx = \sum_1^\infty \int_a^b f_n(x)dx.$$

Proof We showed $s(x)$ is continuous in Section 4.5. If we write

$$s_N(x) = \sum_1^N f_n(x)$$

then we have $s_N(x) \to s(x)$ uniformly on $[a, b]$ and therefore

$$\int_a^b s_N(x)dx \to \int_a^b s(x)dx$$

by Section 4.4. But

$$\int_a^b s_N(x)dx = \sum_1^N \int_a^b f_n(x)dx.$$

Hence

$$\sum_1^\infty \int_a^b f_n(x)dx = \int_a^b s(x)dx$$

as required. Q.E.D.

4.8 **Miscellaneous Exercises** 1. Discuss the uniform convergence of the following series over all real x.

$$\sum_1^\infty \frac{x}{1+n^2x^2} \qquad \sum_1^\infty \frac{x}{1+n^3x^2}$$

$$\sum_1^\infty \sin^n x \cos x \qquad \sum_1^\infty (-1)^n \sin^n x \cos x$$

2. Discuss the continuity of the sums of the following series.

$$\sum_1^\infty \frac{\sin nx}{\sqrt{n}} \qquad \sum_1^\infty \frac{\cos nx}{n\sqrt{n}}$$

$$\sum_1^\infty \frac{n+1}{n+2}\,\frac{\sin nx}{\sqrt{n}} \qquad \sum_1^\infty \log n\,\frac{\sin nx}{\sqrt{n}}$$

3. Prove the following assertions about the series

$$\sum_1^\infty (-1)^n x^n (1-x).$$

(i) It is pointwise absolutely convergent over $0 \le x \le 1$.
(ii) It is uniformly convergent over $0 \le x \le 1$.
(iii) It is **not** uniformly absolutely convergent over $0 \le x \le 1$.

4. Let $f_n(x)$ be the isosceles triangle with base $[1/(n+1), 1/n]$ and height $1/n$, otherwise zero. Show that the series

$$\sum_1^\infty f_n(x)$$

converges uniformly over $0 \le x \le 1$.

Show however that if

$$M_n = \max_{0\le x\le 1} f_n(x)$$

then

$$\sum_1^\infty M_n$$

diverges.

Why does this not contradict the Weierstrass M-Test?