Chapter 3
The Cauchy Criterion

3.0 **Preamble** Having introduced the idea of uniform convergence of sequences, and having mentioned how uniform convergence makes sequential limits behave as they ought to in terms of analytic properties such as continuity, and analytic processes such as differentiation and integration, we are almost ready to move on to the formulation of a theory of uniform convergence for series, which is where the most important applications of uniform covergence are to be found. Almost, but not quite. This is because there is a further prerequisite which has to be discussed first, namely, the Cauchy Criterion, to which the present chapter is devoted.

We shall present the Cauchy Criterion in the context of numerical sequences and series in this chapter, and delay the uniform analogues until next chapter. The Cauchy Criterion is a necessary and sufficient condition for convergence which has the useful feature of not requiring one to know what the limit of the sequence is. This makes it particularly well suited to series where one only rarely knows what the sum of the series is likely to be.

We discuss sequences first, and then graduate to series where the main applications are. Another important feature of the Cauchy Criterion for series is that it makes no demands regarding positivity of terms. This enables tests to be obtained for conditional convergence such as those due to Dirichlet and Abel which are crucial in the development of the theory of power series and Fourier series considered in chapters 5 and 6.

3.1 **Cauchy Sequences** We shall say a real sequence a_n is a **Cauchy sequence** if for any given $\epsilon > 0$ there exists N such that

$$|a_m - a_n| < \epsilon$$

for all $m, n > N$. We shall call this defining condition the **Cauchy Criterion** for reasons which will become apparent in due course. It is equivalent to saying that $a_m - a_n \to 0$ as m, n both $\to \infty$. For examples of Cauchy sequences we need look no further than the following lemma.

Lemma Any convergent sequence is a Cauchy sequence.

Proof Suppose $a_n \to a$ as $n \to \infty$. Then for any given $\epsilon > 0$ there exists N such that

$$|a_n - a| < \epsilon/2$$

for all $n > N$. Therefore for any $m, n > N$ we have

$$\begin{aligned} |a_m - a_n| &\le |a_m - a| + |a - a_n| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence a_n is a Cauchy sequence. Q.E.D.

The remarkable thing is that the converse of this lemma is also true. Not only is the Cauchy Criterion a necessary condition for convergence of a sequence, it is also a sufficient condition. This result was originally proved by Cauchy in the early part of the 19th Century, which is the reason why the condition and type of sequence are named after him.

Informally convergent sequences are those whose terms get close to a definite value, Cauchy sequences are those whose terms get close to each other. It is easy to see that if the terms get close to a limit, then they must get close to each other. What is not so obvious is that if the terms get close to each other then there should be a limit that they get close to. Whilst bees may cluster round a honey-pot it is not at all clear that given a cluster of bees there should be a honey-pot they are clustering round.

Exercises 1. Show that any Cauchy sequence must be bounded.

2. Show that if a Cauchy sequence a_n has a convergent subsequence $a_{n_k} \to a$ then the whole sequence $a_n \to a$.

3.2 **The Cauchy Criterion** We now prove the sufficiency of the Cauchy Criterion for convergence of a real sequence.

Theorem The real sequence a_n converges if and only if given $\epsilon > 0$ there exists N such that

$$|a_m - a_n| < \epsilon$$

for all $m, n > N$.

Proof Suppose a_n is a Cauchy sequence. Define two new sequences b_n, c_n by saying

$$b_n = \inf_{k \geq n} a_k, \quad c_n = \sup_{k \geq n} a_k.$$

Obserne that b_n increases and c_n decreases. We will show these sequences are bounded and therefore convergent. Secondly, we show they have a common limit. Finally we show a_n converges to this common limit.

Suppose $\epsilon > 0$ is given. Then we can choose N such that

$$|a_m - a_n| < \epsilon$$

for all $m, n \geq N$. In particular,

$$a_N - \epsilon < a_n < a_N + \epsilon$$

for all $n \geq N$. Therefore

$$a_N - \epsilon \leq b_n \leq c_n \leq a_N + \epsilon$$

for all $n \geq N$. This shows b_n, c_n are both bounded. (And finite!) It also shows that if $b_n \to b$, $c_n \to c$ then

$$a_N - \epsilon \leq b \leq c \leq a_N + \epsilon,$$

and therefore

$$0 \le c - b \le 2\epsilon.$$

We can now let $\epsilon \to 0$ to obtain $b = c$. Finally, referring back to the definition of b_n, c_n, it is easy to see that

$$b_n \le a_n \le c_n$$

for all n. Hence also $a_n \to b = c$. Q.E.D.

Exercises The Bolzano-Weierstrass Theorem states that any bounded real sequence has a convergent subsequence. Use the Bolzano-Weierstrass Theorem in conjunction with the exercises at the end of Section 3.1 to obtain an alternative proof of Theorem 3.2.

3.3 **Cauchy Criterion for Series** Convergence of an infinite series

$$\sum_1^\infty a_n$$

depends on convergence of the sequence

$$s_N = \sum_1^N a_n$$

of its partial sums. The difference between two terms in the sequence s_N is

$$s_P - s_Q = \sum_{Q+1}^P a_n$$

so the Cauchy Criterion for s_N is that the sum of the block of terms

$$\sum_{Q+1}^P a_n$$

should $\to 0$ as P, Q both $\to \infty$. We call this condition the Cauchy Criterion for the **series**

$$\sum_1^\infty a_n.$$

After a minor modification of the notation we arrive at the following theorem.

Theorem The series

$$\sum_1^\infty a_n$$

converges if and only if given any $\epsilon > 0$ there exists N such that

$$\left| \sum_{P}^{Q} a_n \right| < \epsilon$$

for all $P, Q > N$.

Proof See above. Q.E.D.

As a first application of the Cauchy Criterion for series we present a short proof of the result that absolute convergence implies convergence of a series. In fact, suppose the series

$$\sum_{1}^{\infty} a_n$$

is absolutely convergent, i.e.,

$$\sum_{1}^{\infty} |a_n|$$

converges, and that $\epsilon > 0$ is given. Then by the Cauchy Criterion (necessity of) we can choose N such that

$$\sum_{P}^{Q} |a_n| < \epsilon$$

for all $P, Q > N$. Therefore also

$$\left| \sum_{P}^{Q} a_n \right| \leq \sum_{P}^{Q} |a_n| < \epsilon$$

for all $P, Q > N$. Hence

$$\sum_{1}^{\infty} a_n$$

converges by the Cauchy Criterion (sufficiency of).

Exercises 1. Use the Cauchy Criterion to show the series

$$\sum_{1}^{\infty} \frac{1}{n}$$

diverges.

Hint Show

$$\sum_{N+1}^{2N} \frac{1}{n} \not\to 0.$$

2. Find the value of

$$\lim_{N\to\infty} \sum_{N+1}^{2N} \frac{1}{n}.$$

Hint Consider approximating sums to the integral

$$\int_1^2 \log x dx.$$

3.4 **Tests for Conditional Convergence** For series which fail to converge absolutely there is little in the way of tests available for determining whether they converge or not. The Cauchy Criterion enables this gap to be filled to some extent since it makes no demands regarding positivity of the terms of a series. We present two tests named after Dirichlet and Abel, two eminent mathematicians from the first half of the 19th Century. Dirichlet's test was designed to investigate convergence of trigonometric series, whilst Abel's test has been distilled from his work on the convergence of power series on their circle of convergence.

Theorem Dirichlet's Test The series

$$\sum_1^\infty a_n b_n$$

converges if

(i) there exists M such that

$$\left|\sum_1^N a_n\right| \leq M$$

for all $N \geq 1$,

(ii) $b_n \geq b_{n+1}$ for all $n \geq 1$,

(iii) $b_n \to 0$ as $n \to \infty$.

Proof If we write

$$s_N = \sum_1^N a_n$$

then we have

$$\begin{aligned}\sum_P^Q a_n b_n &= \sum_P^Q (s_n - s_{n-1}) b_n \\ &= -s_{P-1} b_P + \sum_P^{Q-1} s_n (b_n - b_{n+1}) + s_Q b_Q.\end{aligned}$$

Therefore

$$\left|\sum_P^Q a_n b_n\right| \leq M b_P + \sum_P^{Q-1} M(b_n - b_{n+1}) + M b_Q = 2M b_P$$

using conditions (i) and (ii).

Suppose $\epsilon > 0$ is given. Then by condition (iii) we can choose N such that

$$b_n < \epsilon/2M$$

for all $n > N$. Therefore for any $P, Q > N$ we have

$$\left| \sum_P^Q a_n b_n \right| \leq 2M b_P < \epsilon.$$

Hence the series

$$\sum_1^\infty a_n b_n$$

converges by the Cauchy Criterion. Q.E.D.

Corollary Abel's Test The series

$$\sum_1^\infty a_n b_n$$

converges if

(i) the series

$$\sum_1^\infty a_n$$

converges,

(ii) $b_n \geq b_{n+1}$ for all $n \geq 1$,

(iii) b_n is bounded.

Observe Abel's Test makes more demands on a_n, less on b_n.

Proof Conditions (ii) and (iii) imply

$$b = \lim_{n\to\infty} b_n$$

exists. Therefore

$$\sum_1^\infty a_n b_n = b \sum_1^\infty a_n + \sum_1^\infty a_n (b_n - b)$$

is a linear combination of the convergent series

$$\sum_1^\infty a_n$$

by condition (i) and the convergent series

$$\sum_1^\infty a_n(b_n - b)$$

by Dirichlet's Test. Hence

$$\sum_1^\infty a_n b_n$$

converges. Q.E.D.

The application of Dirichlet's and Abel's tests to convergence of trigonometric series is based on the following lemma.

Lemma For all $0 < x < 2\pi$ and all $N \geq 1$

$$\left|\sum_1^N \cos nx\right| \leq \frac{1}{\sin(x/2)}, \qquad \left|\sum_1^N \sin nx\right| \leq \frac{1}{\sin(x/2)}.$$

Proof Using complex numbers we have

$$\left|\sum_1^N \cos nx + i\sum_1^N \sin nx\right| = \left|\sum_1^N e^{inx}\right|$$

$$= \left|\frac{e^{i(N+1)x} - e^{ix}}{e^{ix} - 1}\right|,$$

since we are summing a G.P.,

$$= \left|\frac{e^{i(N+1/2)x} - e^{ix/2}}{e^{ix/2} - e^{-ix/2}}\right|,$$

dividing through by $e^{ix/2}$,

$$\leq \frac{|e^{i(N+1/2)x}| + |e^{ix/2}|}{|2i\sin(x/2)|}$$

$$= \frac{1}{\sin(x/2)}$$

for all $0 < x < 2\pi$ and all $N \geq 1$. Q.E.D.

Example 1 The series

$$\sum_1^\infty \frac{\sin nx}{n}$$

converges for all real x.

Proof The series converges trivially for $x = 2k\pi$. Otherwise use Dirichlet's test with $a_n = \sin nx$, $b_n = 1/n$.

Example 2 The series

$$\sum_1^\infty \frac{\cos nx}{n^2}$$

converges for all real x.

Proof Here we have

$$\left|\frac{\cos nx}{n^2}\right| \leq \frac{1}{n^2}$$

so the series converges absolutely for all real x.

N.B. Dirichlet's and Abel's tests should only be used when tests for absolute convergence have failed.

Exercises For which values of x do the following series converge?

$$\sum_1^\infty \frac{n+1}{n+2}\,\frac{\sin nx}{n} \qquad \sum_1^\infty \log n\,\frac{\cos nx}{n}$$

3.5 **Miscellaneous Exercises on Chapter 3** 1. For which values of x do the series

$$\sum_1^\infty \frac{\cos nx}{n \log n} \qquad \sum_1^\infty \frac{\sin nx}{n(\log n)^2}$$

converge?

2. Show the series

$$\sum_1^\infty \frac{\sin^2 nx}{n}$$

diverges for all $x \neq k\pi$.

Deduce that the series

$$\sum_1^\infty \frac{\sin nx}{n}$$

is not absolutely convergent for $x \neq k\pi$.

3. Discuss convergence of

$$\sum_1^\infty \frac{\sin nx\,\sin ny}{n}$$

for various values of x, y.

4. Show that the condition $b_n \geq b_{n+1}$ in Dirichlet's test can be replaced by

$$\sum_1^\infty |b_n - b_{n+1}| < \infty.$$

How about Abel's test?
5. Which of the following series converge?

$$\sum_1^\infty (-1)^n \frac{\sin \log n}{n} \qquad \sum_1^\infty \frac{\sin \log n}{n}$$

Hint Use Question 4.