# Chapter 2
# Analytic Properties of Uniform Limits

2.1 **Pointwise Limits** Suppose we have a sequence of functions $f_n(x)$ converging to a limit function $f(x)$ pointwise on an interval $I$. We shall be concerned in this chapter with the problem of how analytic properties of the functions $f_n(x)$ carry over to the function $f(x)$. For example, given that $f_n(x)$ is continuous on $I$ for every $n$ can we infer that $f(x)$ is continuous on $I$?

Much as one might like to assume that properties of $f_n(x)$ do carry over to $f(x)$ there is really no reason why they should. For example, if $p_N(x)$ is the polynomial

$$p_N(x) = \sum_0^N \frac{x^n}{n!}$$

(where $0! = 1$) then $p_N(x) \to e^x$ pointwise for all real $x$, but $e^x$ is **not** a polynomial. On a more elementary level, if we have a convergent sequence of rational numbers the limit may well be irrational. In fact the norm would appear to be that if one takes a sequence of objects in a particular system then the limit object in general belongs to some larger system.

The example $f_n(x) = x^n$ which converges pointwise over $-1 < x \leq 1$ to

$$\begin{aligned} f(x) &= 0 \quad (-1 < x < 1) \\ &= 1 \quad (x = 1) \end{aligned}$$

shows immediately that we can have a sequence of continuous functions converging pointwise to a discontinuous limit function. Observe however that this sequence is not uniformly convergent over $-1 < x \leq 1$. We shall show in this chapter that whenever a sequence $f_n(x)$ converges to a limit function uniformly then continuity does carry over from $f_n(x)$ to $f(x)$.

We shall show also that if $f_n(x)$ is continuous and converges uniformly to $f(x)$ over the interval $a \leq x \leq b$ then

$$\int_a^b f_n(x)dx \to \int_a^b f(x)dx.$$

This is another result that is not in general true if we only assume pointwise convergence. (See 2.3)

The situation for carry over of differentiability properties turns out to be slightly more complicated. If $f_n(x) \to f(x)$ uniformly over $a \leq x \leq b$ and $f_n'(x)$ exists over $a \leq x \leq b$ for all $n$ we cannot in general deduce that $f'(x)$ exists or that

$$f_n'(x) \to f'(x).$$

We shall see that these results only hold when the sequence of **derivatives** is assumed to be uniformly convergent. (See 2.4)

**Exercises** 1. Why is $e^x$ not a polynomial? Can you think of a property which polynomials have but which $e^x$ doesn't have? Or vice versa?

2. Let the integers $a_n, b_n$ be defined by the formula

$$(1+\sqrt{2})^n = a_n + b_n\sqrt{2}.$$

Show $r_n = a_n/b_n$ is a sequence of rationals which converges to $\sqrt{2}$.

2.2 **Continuous Functions** The main result of this section is that a uniform limit of continuous functions is continuous. Formally we have the following theorem.

**Theorem** If $f_n(x) \to f(x)$ uniformly on $I$ and if $f_n(x)$ is continuous on $I$ for every $n$ then also $f(x)$ is continuous on $I$.

**Proof** Suppose $a \in I$ and that $\epsilon > 0$ is given. Then because of the uniform convergence there must exist $N$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}$$

for all $n > N$ and all $x \in I$. Choose and fix $n$ satisfying $n > N$. Then for this $n$ there must exist $\delta > 0$ such that

$$|f_n(x) - f_n(a)| < \frac{\epsilon}{3}$$

for all $x \in I$ satisfying $|x - a| < \delta$. Therefore also

$$\begin{aligned} |f(x) - f(a)| &\le |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

for all $x \in I$ satisfying $|x - a| < \delta$. Hence $f(x)$ is continuous at $x = a$ for every $a \in I$. Q.E.D.

**Corollary** If $f_n(x) \to f(x)$ pointwise over $I$ and $f_n(x)$ is continuous on $I$ for all $n$ but $f(x)$ is **not** continuous on $I$ then the convergence cannot be uniform over $I$.

**Proof** If the convergence were uniform then the theorem would be contradicted. Q.E.D.

The above corollary gives an alternative proof of the non-uniform convergence of e.g. $x^n$ over $-1 < x \le 1$ since the limit function

$$\begin{aligned} f(x) &= 0 \quad (-1 < x < 1) \\ &= 1 \quad (x = 1) \end{aligned}$$

has a discontinuity at $x = 1$.

Continuity of the limit function does **not** imply that the convergence is uniform. A pointwise limit of a sequence of continuous functions may be continuous even when the convergence is not uniform. Consider for example the sequence

$$f_n(x) = nxe^{-nx}$$

which converges pointwise to $f(x) \equiv 0$ over $x \geq 0$. By calculus we have

$$\begin{aligned} M_n &= \sup_{x \geq 0} |f_n(x) - f(x)| \\ &= \sup_{x \geq 0} nxe^{-nx} \\ &= 1/e \end{aligned}$$

attained at $x = 1/n$. Therefore $M_n \not\to 0$ and hence $f_n(x)$ is not uniformly covergent over $x \geq 0$.

**Exercises** Comment on the uniform convergence and continuity of the limit function of the following sequences.

(i) $f_n(x)$ is the function whose graph is the isosceles triangle with base $-1/n \leq x \leq 1/n$ and with height 1, otherwise zero.

(ii) Ditto but with base $0 \leq x \leq 2/n$.

Draw graphs.

2.3 **Integration of Uniformly Convergent Sequences** The question we address in this section is, given that $f_n(x) \to f(x)$ pointwise over an interval $I$, when can one say

$$\int_I f_n(x)dx \to \int_I f(x)dx?$$

We shall show the question has an affirmative answer when $I$ is a closed bounded interval and $f_n(x) \to f(x)$ uniformly over $I$.

The example $f_n(x)$ having graph the isosceles triangle with base $0 \leq x \leq 1/n$ and height $n$ (otherwise zero) shows the answer may be negative in the case of non-uniform convergence. In fact, for this $f_n(x)$ we have $f_n(x) \to f(x) \equiv 0$ pointwise over all real $x$, since for $x > 0$ we have $f_n(x) = 0$ for all large $n$ ($> 1/x$), and $f_n(0) = 0$ for all $n$, but e.g.

$$\int_0^1 f_n(x)dx = 1/2 \not\to 0 = \int_0^1 f(x)dx.$$

The answer may also be negative in the case when the interval of integration is unbounded. Consider for example $f_n(x)$ whose graph is the isosceles triangle with base $0 \leq x \leq n$ and height $1/n$. Again $f_n(x) \to f(x) \equiv 0$ pointwise, in this case uniformly, over all real $x$ but

$$\int_{-\infty}^{\infty} f_n(x)dx = 1/2 \not\to 0 = \int_{-\infty}^{\infty} f(x)dx.$$

We shall restrict attention to continuous functions for the time being. Positive results exist for more general functions but we shall delay consideration of these until a later stage.

**Theorem** If $f_n(x) \to f(x)$ uniformly over the bounded closed interval $[a,b]$ and if $f_n(x)$ is continuous on $[a,b]$ for every $n$ then

$$\int_a^b f_n(x)dx \to \int_a^b f(x)dx.$$

**Proof** By Theorem 2.2 the limit function $f(x)$ is continuous so

$$\int_a^b f(x)dx$$

certainly exists. Suppose $\epsilon > 0$ is given. Then we can choose $N$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$$

for all $n > N$ and all $a \le x \le b$. Therefore for all $n > N$ we have

$$\begin{aligned}\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| &= \left|\int_a^b \big(f_n(x) - f(x)\big)dx\right| \\ &\le \int_a^b \big|f_n(x) - f(x)\big|dx \\ &< \int_a^b \frac{\epsilon}{b-a}dx \\ &= \epsilon.\end{aligned}$$

Hence

$$\int_a^b f_n(x)dx \to \int_a^b f(x)dx$$

as required. Q.E.D.

**Exercises** Find

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx$$

for the following $f_n(x)$.

$$\text{(i)} \quad n^2x^n(1-x) \qquad \text{(ii)} \quad x^n(1-x)^n$$

Comment on uniform convergence and consistency with the theorem of this section.

2.4 **Differentiation of Uniformly Convergent Sequences** We now wish to investigate conditions under which, given $f_n(x) \to f(x)$ pointwise on an interval $I$, we can infer that also $f_n'(x) \to f'(x)$ pointwise on $I$. We shall have to assume $f_n'(x)$ exists on $I$ for every $n$ of course. The natural conjecture is that if also $f_n(x) \to f(x)$ uniformly on $I$ then $f'(x)$ must exist on $I$ and be the pointwise (possibly uniform?) limit of $f_n'(x)$ on $I$. Regrettably this conjecture turns out to be false as the following example shows.

Consider $f_n(x) = xe^{-nx}$. We have $f_n(x) \to f(x) \equiv 0$ uniformly over $x \geq 0$. The differentiated sequence is

$$f_n'(x) = (1 - nx)e^{-nx}$$

whose pointwise limit over $x \geq 0$ is $g(x)$ where

$$\begin{aligned} g(x) &= 1 \quad (x = 0) \\ &= 0 \quad (x > 0) \end{aligned}$$

which is $\neq f'(x)$ at $x = 0$.

The only way to get a positive result is to make demands on the differentiated sequence. The key condition is to require that $f_n'(x)$ be uniformly convergent on $I$. Uniform convergence of the sequence $f_n(x)$ turns out to be immaterial. Explicitly we have the following theorem.

**Theorem** If $f_n(x) \to f(x)$ pointwise on an interval $I$ and if $f_n'(x)$ exists and $\to g(x)$ uniformly on $I$ then $f'(x)$ exists $= g(x)$ on $I$.

**Proof** We shall assume $f_n'(x)$ is continuous on $I$ for all $n$. This will be sufficient for all the applications we have in mind, though the theorem can be proved without this assumption. With the benefit of this extra assumption it follows by Theorem 2.2 that $g(x)$ is continuous on $I$, and by Theorem 2.3 that

$$\int_a^x f_n'(t)dt \to \int_a^x g(t)dt$$

for any $a, x \in I$. But by the Fundamental Theorem of Calculus (integration of derivative form)

$$\begin{aligned} \int_a^x f_n'(t)dt &= f_n(x) - f_n(a) \\ &\to f(x) - f(a) \end{aligned}$$

as $n \to \infty$. Therefore

$$\int_a^x g(t)dt = f(x) - f(a)$$

which on rearranging gives

$$f(x) = f(a) + \int_a^x g(t)dt.$$

Hence by the Fundamental Theorem of Calculus again (differentiation of integral form) $f'(x)$ exists $= g(x)$ for all $x \in I$ as required. Q.E.D.

Exercises Investigate convergence of $f_n'(x)$ for the following sequences $f_n(x)$.

$$\text{(i)} \quad \frac{x^n}{n} \quad \text{over} \quad |x| \le 1,$$
$$\text{(ii)} \quad \frac{x}{1+n^2x^2} \quad \text{over all real } x.$$

Draw graphs.

2.5 **Almost Uniform Convergence** Consider the example $f_n(x) = x^n$. The pointwise limit on the interval $I = (-1, 1)$ is $f(x) \equiv 0$. The convergence is not uniform on $I$ since

$$M_n = \sup_{x \in I} |f_n(x) - f(x)| = \sup_{|x|<1} |x^n| = 1 \not\to 0$$

as $n \to \infty$. However if $J$ is any **closed** subinterval of $I$ then the convergence **is** uniform on $J$. This is because there must exist $r < 1$ such that $J \subseteq [-r, r]$ and therefore

$$M_n = \sup_{x \in J} |f_n(x) - f(x)| = \sup_{x \in J} |x^n| \le \sup_{|x| \le r} |x^n| = r^n \to 0$$

as $n \to \infty$. This is an example of 'almost' uniform convergence which we now formally define.

**Definition** We say $f_n(x) \to f(x)$ **almost uniformly** on an open interval $I$ if $f_n(x) \to f(x)$ uniformly on every closed subinterval $J \subseteq I$.

The usefulness of almost uniform convergence is that in many applications it is just as good as uniform convergence. For example, if $f_n(x) \to f(x)$ almost uniformly on an open interval $I$ and $f_n(x)$ is continuous on $I$ for every $n$ then also $f(x)$ must be continuous on $I$. To see this, observe that for any $a \in I$ it is possible to choose a closed subinterval $J \subseteq I$ such that $a \in J$. Therefore $f_n(x) \to f(x)$ uniformly on $J$ so, by Theorem 2.2, $f(x)$ must be continuous on $J$. In particular, $f(x)$ must be continuous at $a$.

**Exercises** 1. Show that if $f_n(x) \to f(x)$ almost uniformly on an open interval $I$ and if $f_n(x)$ is continuous on $I$ for all $n$ then

$$\int_x^y f_n(t)dt \to \int_x^y f(t)dt$$

for any $x, y \in I$.

2. Show that if $f_n(x) \to f(x)$ pointwise on an open interval $I$ and if $f_n'(x)$ exists and $\to g(x)$ almost uniformly on $I$ then $f'(x)$ exists $= g(x)$ on $I$.

2.6 **Miscellaneous Exercises** 1. Show the following sequences are uniformly convergent over $0 \le x \le A$ for any fixed $A > 0$, but not uniformly convergent over $x \ge 0$.

$$\text{(i)} \quad \left(\frac{x}{x+1}\right)^n \qquad \text{(ii)} \quad \frac{nx}{x^2+n^2}$$

2. Show the following sequences are uniformly convergent over $x \geq \epsilon$ for any fixed $\epsilon > 0$, but not uniformly convergent over $x > 0$.

$$\text{(i)} \quad \left(\frac{1}{x+1}\right)^n \qquad \text{(ii)} \quad \frac{n^2x^2}{(1+n^2x^2)^2}$$

3. Show that if $f(t)$ is continuous for $t \geq 0$ and $x_n \to x$ (all $\geq 0$) then for any fixed $A \geq 0$

$$\int_0^A e^{-x_n t} f(t)dt \to \int_0^A e^{-xt} f(t)dt.$$

**Hint** Use the inequality $|e^{-x} - e^{-y}| \leq |x - y|$ for $x, y \geq 0$.

4. Show that if $f(t)$ is continuous for $t \geq 0$ and

$$\int_0^\infty |f(t)|dt$$

exists then

$$\int_A^\infty e^{-xt} f(t)dt \to 0$$

as $A \to \infty$ uniformly over $x \geq 0$.

5. Use Questions 3 and 4 to show that if $f(t)$ is continuous for $t \geq 0$ and

$$\int_0^\infty |f(t)|dt$$

exists then the Laplace Transform

$$F(x) = \int_0^\infty e^{-xt} f(t)dt$$

of $f(t)$ is continuous for $x \geq 0$.