Chapter 1
Definition of Uniform Convergence

1.1 **Motivation** A typical situation where the methods of elementary analysis are inadequate is the following. Suppose one tries to solve the differential equation

$$\frac{dy}{dx} = y$$

by substituting for y the power series

$$y = \sum_0^\infty a_n x^n.$$

Differentiating the power series term by term gives

$$\frac{dy}{dx} = \sum_1^\infty n a_n x^{n-1} = \sum_0^\infty (n+1) a_{n+1} x^n,$$

which on substituting in the differential equation and equating coefficients gives

$$(n+1)a_{n+1} = a_n$$

for all $n \geq 0$. If we put $a_0 = A$ then we get $a_n = A/n!$ for all $n \geq 0$ and hence the solution

$$y = A \sum_0^\infty \frac{x^n}{n!} = Ae^x$$

as expected.

Any attempt to prove rigorously that this approach does indeed give rise to a genuine solution of the differential equation would have to include a demonstration of the validity of differentiating an infinite series term by term. Whereas the only general theorem available in elementary analysis about differentiating sums of functions is for **finite** sums only. In fact it is in general not possible to differentiate an **infinite** series term by term as the following counter-example shows.

Consider the series

$$\sum_1^\infty \frac{\sin nx}{n} = \frac{\pi - x}{2}$$

valid for $0 < x < 2\pi$. Differentiation term by term gives

$$\sum_1^\infty \cos nx = -\frac{1}{2}$$

which is clearly false for e.g. $x = \pi/2$. Observe also that the sum of the series

$$s(x) = \sum_1^\infty \frac{\sin nx}{n}$$

has discontinuities at $x = 0, 2\pi$ in spite of the fact that the nth term

$$\frac{\sin nx}{n}$$

is a continuous function of x for all $n \geq 1$. Observe further however that if we **integrate** the above series term by term over the interval $0 \leq x \leq \pi$ we get

$$\sum_1^\infty \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

which is **true**.

Elementary analysis does not allow us to infer that the sum of an infinite series of continuous functions is another continuous function, or that we can integrate an infinite series term by term. If we wish to make such inferences we have to impose extra conditions on the series involved, and this is where uniform convergence comes in. Uniform convergence is a special kind of convergence which enables us to justify differentiating and integrating series term by term. It also enables us to show that if all the terms of a series are continuous functions then so is the sum.

We shall spend this chapter describing what uniform convergence is. The description will be in the context of sequences rather than series in the first instance. We give three equivalent definitions of uniform convergence of a sequence of functions. The first definition is in terms of epsilontics and explains the basic idea. The second definition takes the form of an algebraic criterion for uniform convergence of a sequence and gives an easy method for testing whether a given sequence is uniformly convergent or not. The third definition is geometric and provides the increased insight that a visual approach always brings to a mathematical investigation.

Exercises 1. Solve

$$\frac{d^2y}{dx^2} = -y$$

by the method given above.

2. Show that if $|x| \leq \pi/2$ then $|m\pi - nx| \leq \pi/4$ for infinitely many integers m, n. Deduce that $\cos nx \not\to 0$ as $n \to \infty$ and hence that the series

$$\sum_1^\infty \cos nx$$

diverges for every x.

3. Use the formula

$$\sum_1^\infty \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

to obtain

$$\text{(i)} \quad \sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \text{(ii)} \quad \sum_1^\infty \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

1.2 **Convergence with a Parameter** Consider the sequence whose nth term is x^n where x is real. From elementary analysis we know that $x^n \to 0$ as $n \to \infty$ whenever $|x| < 1$. In this type of situation we shall call x a **parameter**, by which we mean a higher order of variable which remains fixed whilst lower order variables vary, in this case n.

To verify that $x^n \to 0$ in terms of epsilontics we have to show that for any given $\epsilon > 0$ there exists N such that

$$|x^n| < \epsilon$$

for all $n > N$. Now the inequality

$$|x^n| < \epsilon$$

is equivalent to

$$n \log|x| < \log \epsilon,$$

and therefore to

$$n > \frac{\log \epsilon}{\log|x|},$$

bearing in mind that $\log|x| < 0$ for $|x| < 1$. So the required N is

$$N = \frac{\log \epsilon}{\log|x|},$$

or the smallest integer greater than this number if one insists that N should itself be an integer. Observe that N depends on the parameter x as well as the given ϵ.

The epsilontic approach to convergence of a sequence can be regarded as a game played between two players one choosing ϵ and the other choosing N. The ϵ-player goes first and the N-player follows. The N-player wins if he can find N such that the nth term of the sequence is within ϵ of its limit for all $n > N$. The sequence converges when the N-player can win whatever ϵ the ϵ-player chooses. For the sequence x^n with $|x| < 1$ the N-player's winning strategy is to choose

$$N = \frac{\log \epsilon}{\log|x|}.$$

For convergence with a parameter the game is three-handed. In the case of the sequence x^n we have an x-player as well as an ϵ-player and N-player. The order of play is x first, then ϵ and then N. The N-player must wait until both ϵ, x have been specified before he can play because the expression

$$\frac{\log \epsilon}{\log|x|}$$

can be arbitrarily large for ϵ small or $|x|$ near to 1.

Now consider instead the sequence x^n/n. This sequence converges to zero for any $|x| \leq 1$ since

$$\left|\frac{x^n}{n}\right| \leq \frac{1}{n} < \epsilon$$

for all $|x| \leq 1$ provided $n > 1/\epsilon$. If we take $N = 1/\epsilon$ then we have

$$\left|\frac{x^n}{n}\right| < \epsilon$$

for all $n > N$ and all $|x| \leq 1$. Observe that, for this sequence, N is **independent** of the parameter x. In terms of the convergence game described above the N-player can still win if the x-player plays **last** rather than first.

The requirement that N be chosen **before** x is the essential feature of uniform convergence which we define formally in the next section. We shall see that of the two sequences considered above the second one is uniformly convergent whilst the first one isn't.

Exercises 1. Find N in terms of $\epsilon > 0, \alpha > 0$ such that

$$\frac{1}{n^\alpha} < \epsilon$$

for all $n > N$. Can N be chosen independently of α?

2. Find N in terms of $\epsilon > 0, \theta$ real such that

$$\left|\frac{\sin n\theta}{n}\right| < \epsilon$$

for all $n > N$. Can N be chosen independently of θ?

1.3 **Definition of Uniform Convergence** Suppose that $(f_n)_{n\geq 1}$ is a sequence of real-valued functions and that f is another real-valued function all defined on the same real interval I. Then we shall say that $f_n \to f$ **pointwise** on I if for each point $x \in I$ the numerical sequence $f_n(x) \to f(x)$ as $n \to \infty$. For example if I is the interval $-1 < x \leq 1$ and $f_n(x) = x^n$ then $f_n \to f$ pointwise on I where

$$\begin{aligned} f(x) &= 0 \quad (-1 < x < 1), \\ &= 1 \quad (x = 1). \end{aligned}$$

We shall call f the **pointwise limit function** of the sequence $(f_n)_{n\geq 1}$.

Epsilontically speaking, $f_n \to f$ pointwise on I if given any $x \in I$ and any $\epsilon > 0$ there exists N such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $n > N$. We shall say $f_n \to f$ **uniformly** on I if given $\epsilon > 0$ there exists N such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $n > N$ and for all $x \in I$.

The crucial difference between pointwise and uniform convergence is the order in which the variables ϵ, N, x appear. For pointwise convergence N is chosen last and is allowed to depend on both ϵ and x. For uniform convergence N must be chosen immediately after ϵ has been specified and must do duty for all x simultaneously. This means N depends only on ϵ and must be **independent** of x.

Consider again the sequence $f_n(x) = x^n$ which converges pointwise on the interval $I = (-1, 1]$ to

$$\begin{aligned} f(x) &= 0 \quad (-1 < x < 1), \\ &= 1 \quad (x = 1). \end{aligned}$$

Does $f_n \to f$ uniformly over I? We have

$$\begin{aligned} |f_n(x) - f(x)| &= |x|^n \quad (-1 < x < 1), \\ &= 0 \qquad (x = 1) \end{aligned}$$

so the question is whether given $\epsilon > 0$ we can choose N such that

$$|x|^n < \epsilon$$

for all $n > N$ and all $-1 < x < 1$. In fact this is impossible since for **fixed** n the function $|x|^n \to 1$ as $x \to \pm 1$ therefore is e.g. $> 1/2$ for some x satisfying $|x| < 1$. So if we take $\epsilon = 1/2$ then for any choice of N there will be $n > N$, $|x| < 1$ such that

$$|x|^n > \epsilon.$$

Hence $f_n \not\to f$ uniformly over I in this case.

However if $(f_n)_{n \geq 1}$ is the sequence

$$f_n(x) = \frac{x^n}{n}$$

then we have the pointwise limit $f(x) \equiv 0$ over the interval $I = [-1, 1]$. Therefore

$$|f_n(x) - f(x)| = \frac{|x^n|}{n} \leq \frac{1}{n}$$

for all $n \geq 1$ and all $x \in I$. So if $\epsilon > 0$ is given we can choose N such that

$$\frac{1}{n} < \epsilon$$

for all $n > N$. For this N we have

$$|f_n(x) - f(x)| < \epsilon$$

for all $n > N$ and $x \in I$. Hence for this sequence we do have $f_n \to f$ uniformly over I.

Exercises 1. Does the sequence $1/n^\alpha \to 0$ as $n \to \infty$ uniformly over $\alpha > 0$?
2. Does the sequence

$$\frac{\sin n\theta}{n} \to 0$$

as $n \to \infty$ uniformly over all real θ?

1.4 **The M-Criterion** A systematic method for deciding whether a given sequence of functions is uniformly convergent over a given interval is provided by the following Theorem.

Theorem If $f_n(x) \to f(x)$ pointwise over I and if

$$M_n = \sup_{x \in I} |f_n(x) - f(x)|$$

then $f_n(x) \to f(x)$ uniformly over I if and only if $M_n \to 0$ as $n \to \infty$.

Proof Suppose that $f_n(x) \to f(x)$ uniformly over I and that $\epsilon > 0$ is given. Then there exists N such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $n > N$ and all $x \in I$. Therefore

$$M_n \leq \epsilon$$

for all $n > N$ and hence $M_n \to 0$.

On the other hand, suppose that $M_n \to 0$ and that $\epsilon > 0$ is given. Then there exists N such that

$$M_n < \epsilon$$

for all $n > N$. Therefore

$$|f_n(x) - f(x)| \leq M_n < \epsilon$$

for all $x \in I$ and all $n > N$. Hence $f_n(x) \to f(x)$ uniformly over I. Q.E.D.

By the 'M-Criterion' we mean the condition that $M_n \to 0$. We call it a criterion because it is a necessary and sufficient condition. The procedure for testing a sequence $f_n(x)$ for uniform convergence with the M-Criterion is firstly to find the pointwise limit $f(x)$, secondly to calculate

$$M_n = \sup_{x \in I} |f_n(x) - f(x)|$$

where I is the interval in question, and finally to decide whether $M_n \to 0$ or not.

Consider for example the sequence $f_n(x) = x^n$. The pointwise limit over the interval $I = (-1, 1]$ is $f(x)$ where

$$\begin{aligned} f(x) &= 0 \quad (-1 < x < 1), \\ &= 1 \quad (x = 1). \end{aligned}$$

Therefore

$$\begin{aligned} M_n &= \sup_{x \in I} |f_n(x) - f(x)| \\ &= \sup_{-1<x<1} |x^n| \\ &= 1 \end{aligned}$$

for all n. In this case $M_n \not\to 0$ as $n \to \infty$ so $f_n(x)$ is not uniformly convergent over I.

However, if $f_n(x) = x^n/n$ then the pointwise limit over $I = [-1, 1]$ is $f(x) \equiv 0$, therefore

$$\begin{aligned} M_n &= \sup_{x \in I} |f_n(x) - f(x)| \\ &= \max_{|x| \le 1} |x^n/n| \\ &= 1/n \end{aligned}$$

which does $\to 0$ as $n \to \infty$, so this time we do have uniform convergence over I.

It is often necessary to use calculus to find the value of M_n. For example, suppose $f_n(x) = xe^{-nx}$. The pointwise limit for $x \ge 0$ is $f(x) \equiv 0$. Therefore

$$M_n = \sup_{x \ge 0} |f_n(x) - f(x)| = \sup_{x \ge 0} xe^{-nx}.$$

Differentiating xe^{-nx} and putting the derivative equal to zero we obtain

$$(1 - nx)e^{-nx} = 0.$$

Therefore the maximum value of xe^{-nx} over $x \ge 0$ is attained at $x = 1/n$ and is

$$M_n = \frac{1}{ne}.$$

Hence $M_n \to 0$ and the sequence is uniformly convergent over $x \ge 0$.

Exercises Use the M-Criterion to decide whether the following sequences are uniformly convergent over $x \ge 0$.

$$\text{(i)} \quad x^2 e^{-nx} \qquad \text{(ii)} \quad e^{-nx} \sin nx$$

1.5 **Geometric Interpretation** Referring back to the definition of uniform convergence, we have $f_n(x) \to f(x)$ uniformly on an interval I if for any given $\epsilon > 0$ there exists N such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $n > N$ and all $x \in I$. We can rephrase the above inequality as

$$f(x) - \epsilon < f_n(x) < f(x) + \epsilon$$

which says geometrically that the graph of $f_n(x)$ must lie between the graphs of $f(x) \pm \epsilon$. We can think of the graphs of $f(x) \pm \epsilon$ as fences erected on either side of the graph of $f(x)$ at a distance ϵ away from it. For uniform convergence we must allow the fences to be chosen arbitrarily close to the graph of $f(x)$ and then show there is an N such that the graph of $f_n(x)$ lies between the fences for all $n > N$.

Consider again the example $f_n(x) = x^n$, pointwise convergent to

$$\begin{aligned} f(x) &= 0 \quad (-1 < x < 1) \\ &= 1 \quad (x = 1) \end{aligned}$$

over the interval $-1 < x \leq 1$. If we take $\epsilon < 1$ then, for every n, the graph of x^n must go outside the region between the graphs of $f(x) \pm \epsilon$ for some x in $|x| < 1$. Which confirms that this sequence is not uniformly convergent over $-1 < x \leq 1$.

However for the example $f_n(x) = x^n/n$, pointwise convergent to $f(x) \equiv 0$ over $|x| \leq 1$, the graph of $f_n(x)$ lies between the graphs of $f(x) \pm 1/n$ for all n. Therefore for given $\epsilon > 0$ the graph of $f_n(x)$ lies between the graphs of $f(x) \pm \epsilon$ for all $1/n < \epsilon$ equivalently $n > N = 1/\epsilon$. Confirming that this sequence is uniformly convergent over $|x| \leq 1$.

Exercises Draw graphs of the functions in the exercises of Section 1.4 and verify uniform or non-uniform convergence by the geometric method described above.

1.6 **Miscellaneous Exercises** 1. Which of the following sequences are uniformly convergent over $0 \leq x \leq 1$?

$$\begin{array}{llllll} \text{(i)} & xe^{-nx^2} & \text{(ii)} & \sin^n \pi x & \text{(iii)} & x^n(1-x) \\ \text{(iv)} & \cos \dfrac{x}{n} & \text{(v)} & \dfrac{nx}{1+n^2x^2} & \text{(vi)} & \left(1+\dfrac{x}{n}\right)^n \end{array}$$

2. Show that if $f_n \to f$ uniformly on the interval I then also $f_n \to f$ on any subinterval $J \subseteq I$.

3. Show that if $f_n \to f$ uniformly on the open interval (a, b) and if $f_n(a) \to f(a)$, $f_n(b) \to f(b)$ then also $f_n \to f$ on the closed interval $[a, b]$.

4. Show that if $f_n \to f$ and $g_n \to g$ uniformly on I and if α, β are constants then also

$$\alpha f_n(x) + \beta g_n(x) \to \alpha f(x) + \beta g(x)$$

uniformly on I.

5. Show that if $f_n \to f$ uniformly on I and if g is bounded on I then also

$$f_n(x)g(x) \to f(x)g(x)$$

uniformly on I.