Appendix

Fourier Series of Continuous Functions As promised in Chapter 6 we now give an example of a continuous function with divergent Fourier series.

For each $n \geq 2$ let $\phi_n(x)$ be

$$\begin{aligned} \phi_n(x) &= \sin n!x \qquad &&\text{if } \pi/n! \leq x \leq \pi/(n-1)! \\ &= 0 &&\text{otherwise,} \end{aligned}$$

and let

$$\phi(x) = \sum_2^\infty c_n \phi_n(x)$$

where c_n is a decreasing sequence which $\to 0$ but such that $c_n \log n \to \infty$. For example, take $c_n = 1/\log\log n$.

The function $\phi(x)$ is clearly continuous since the series is uniformly convergent. In fact, if $c_n < \epsilon$ for $n > N$ then

$$\left| \sum_p^q c_n \phi_n(x) \right| \leq c_p < \epsilon$$

for $q \geq p > N$. Observe that the Weierstrass M-Test is inapplicable here.

We shall show that the Fourier series of $\phi(x)$ diverges at $x = 0$. To achieve this we have to show that the sequence

$$s_N(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(x) \, \frac{\sin(N+1/2)x}{\sin x/2} \, dx$$

diverges as $N \to \infty$. It simplifies the analysis considerably if we observe that

$$s_N(0) \sim \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(x) \, \frac{\sin Nx}{x} \, dx$$

in the sense that

$$s_N(0) - \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(x) \, \frac{\sin Nx}{x} \, dx \to 0$$

as $N \to \infty$. In fact,

$$\frac{\sin(N+1/2)x}{\sin x/2} - \frac{\sin Nx}{x/2} = \sin Nx \,(\cot x/2 - 2/x) + \cos Nx$$

and $\cot x/2 - 2/x \to 0$ as $x \to 0$, so

$$\begin{aligned}
&\int_{-\pi}^{\pi} \phi(x)\, \frac{\sin(N+1/2)x}{\sin x/2}\, dx - \int_{-\pi}^{\pi} \phi(x)\, \frac{\sin Nx}{x/2}\, dx \\
&= \int_{-\pi}^{\pi} \phi(x)\, (\cot x/2 - 2/x) \sin Nx\, dx + \int_{-\pi}^{\pi} \phi(x) \cos Nx\, dx \\
&\to 0
\end{aligned}$$

as $N \to \infty$ since both terms are Fourier coefficients of continuous functions.

Now consider

$$\begin{aligned}
s_{N!}(0) &\sim \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(x)\, \frac{\sin N!x}{x}\, dx \\
&= \sum_{2}^{\infty} \frac{c_n}{\pi} \int_{\pi/n!}^{\pi/(n-1)!} \frac{\sin n!x \sin N!x}{x}\, dx \\
&= \sum_{2}^{N-1} + \frac{c_N}{\pi} \int_{\pi/N!}^{\pi/(N-1)!} \frac{\sin^2 N!x}{x}\, dx + \sum_{N+1}^{\infty}.
\end{aligned}$$

We will show that the middle term $\to \infty$, and that the first and third terms both $\to 0$ as $N \to \infty$.

Regarding the middle term, we have

$$\sin^2 N!x = \frac{1 - \cos 2N!x}{2},$$

so we obtain two integrals, firstly,

$$\frac{c_N}{2\pi} \int_{\pi/N!}^{\pi/(N-1)!} \frac{dx}{x} = \frac{c_N \log N}{2\pi} \to \infty$$

as $N \to \infty$, and, secondly, after integrating by parts,

$$\begin{aligned}
\left| \frac{c_N}{4\pi N!} \int_{\pi/N!}^{\pi/(N-1)!} \frac{\sin 2N!x}{x^2}\, dx \right| &\le \frac{c_N}{4\pi N!} \int_{\pi/N!}^{\pi/(N-1)!} \frac{dx}{x^2} \\
&= \frac{c_N}{4\pi^2}\, \frac{N! - (N-1)!}{N!} \\
&< \frac{c_N}{4\pi^2}.
\end{aligned}$$

Regarding the first term, we have

$$
\begin{aligned}
&\sum_{2}^{N-1} \frac{c_n}{\pi} \int_{\pi/n!}^{\pi/(n-1)!} \frac{\sin n!x \sin N!x}{x}\, dx \\
&= \sum_{2}^{N-1} \frac{c_n}{2\pi} \int_{\pi/n!}^{\pi/(n-1)!} \frac{\cos(N! - n!)x - \cos(N! + n!)x}{x}\, dx \\
&= \sum_{2}^{N-1} \frac{c_n}{2\pi} \frac{1}{N! - n!} \int_{\pi/n!}^{\pi/(n-1)!} \frac{\sin(N! - n!)x}{x^2}\, dx \\
&\quad - \sum_{2}^{N-1} \frac{c_n}{2\pi} \frac{1}{N! + n!} \int_{\pi/n!}^{\pi/(n-1)!} \frac{\sin(N! + n!)x}{x^2}\, dx.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\left| \sum_{2}^{N-1} \right| &\le \frac{c_2}{\pi^2} \sum_{2}^{N-1} \frac{n! - (n-1)!}{N! - (N-1)!} \\
&\le \frac{c_2}{\pi^2} \frac{(N-1)! - 1}{N! - (N-1)!} \\
&\to 0
\end{aligned}
$$

as $N \to \infty$.

Finally, regarding the third term, we have

$$
\begin{aligned}
\left| \sum_{N+1}^{\infty} \frac{c_n}{\pi} \int_{\pi/n!}^{\pi/(n-1)!} \frac{\sin n!x \sin N!x}{x}\, dx \right| &\le \sum_{N+1}^{\infty} \frac{c_n}{\pi} \int_{\pi/n!}^{\pi/(n-1)!} \left| \frac{\sin N!x}{x} \right| dx \\
&\le \sum_{N+1}^{\infty} \frac{c_N}{\pi} \int_{\pi/n!}^{\pi/(n-1)!} N!\, dx \\
&= \frac{c_N}{\pi} \int_{0}^{\pi/N!} N!\, dx \\
&= c_N \\
&\to 0
\end{aligned}
$$

as $N \to \infty$.