# Q29: Prove that an infinite product equals $\sqrt{2}$

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## 1 The problems

Prove that

$$\prod_{k=1}^{\infty}\left(1+\frac{(-1)^{k+1}}{2k-1}\right) = \sqrt{2}.$$

More generally show that

$$\lim_{k\to\infty}\frac{2\,.\,4\,.\,6\,.\,8\,.\,.\,.\,.\,.\,2kN}{1\,.\,3\,.\,5\,.\,.\,.\,.\,.\,(2kN-1)}\cdot\frac{1\,.\,3\,.\,5\,.\,.\,.\,.(2k-1)}{2\,.\,4\,.\,6\,.\,8\,.\,.\,.\,.\,.\,2k} = \sqrt{N}.$$

The infinite product is on page 12 of the tables of series and products by Gradshteyn and Ryzhik who attribute it to Euler in his astounding two-volume 'Introductio in Analysin Infinitorum', 1748. The second is problem 10 on page 104 of the book on infinite series by T. J. l'A Bromwich, 1907.

## 2 The infinite product

I have not looked at the Latin text to see how Euler came by this formula. My solution is less than rigorous since I prove that the logarithm of the product approaches $\frac{1}{2}\ln 2 = \ln\sqrt{2}$ as $k \to \infty$ using Stirling's asymptotic series of the factorial function. The first step is to expand a few factors:

$$\begin{aligned}\Pi &= \left(1+\frac{1}{1}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1+\frac{1}{9}\right)\left(1-\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\ldots.\\ &= \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{10}{9}\cdot\frac{10}{11}\cdot\frac{14}{13}\cdot\frac{14}{15}\ldots\ldots.\\ &= \frac{2^2}{1\,.\,3}\cdot\frac{2^2\,.\,3^2}{5\,.\,7}\cdot\frac{2^2\,.\,5^2}{9\,.\,11}\cdot\frac{2^2\,.\,7^2}{13\,.\,15}\cdot\frac{2^2\,.\,9^2}{17\,.\,19}\ldots..\end{aligned}$$

If this product is truncated at these five double-integer factors labelled 0 to 4,

$$\Pi_4 = 2^{10}\cdot\frac{1^2}{1\,.\,3}\cdot\frac{3^2}{5\,.\,7}\cdot\frac{5^2}{9\,.\,11}\cdot\frac{7^2}{13\,.\,15}\cdot\frac{9^2}{17\,.\,19}\,.$$

The denominator can be written as the quotient of two factorials:

$$1\,.\,3\,.\,4\,.\,5\,.\,7\,.\,9\,.\,\ldots(2n+1) = \frac{(2n+1)!}{2^n\,n!}$$

and similarly the numerator is the squared ratio of factorials, though with a more slowly advancing index. The general product $\Pi_n$ is

$$\Pi_n = \frac{2^{2n+3}\,[(2n+1)!]^3}{(n!)^2\,(4n+3)!}\,. \tag{1}$$

The correspondence between $k$ in the original product and $n$ is $k = 2n+2$. Both forms of the product converge slowly from below as these numerical values illustrate:

$$\Pi_4 = \frac{64512}{46189} \approx 1\cdot 397, \qquad \Pi_{10} \approx 1\cdot 406, \qquad \Pi_{50} \approx 1\cdot 4125, \qquad \Pi_{100} \approx 1\cdot 4133, \qquad \sqrt{2} = 1\cdot 4142....$$

Now I take the logarithm of $\Pi_n$

$$\ln \Pi_n \;=\; (2n+3)\ln 2 \;+\; 3\ln(2n+1)! \;-\; 2\ln n! \;-\; \ln(4n+3)!\,.$$

So far there is no approximation, but now replace the factorials with their asymptotic expansions $\ln n! \sim n\ln n - n + \ln\sqrt{2\pi n} \approx (n+\frac{1}{2})\ln n - n$ as $n \to \infty$.

$$\ln \Pi_n \;\sim\; (2n+3)\ln 2 + 3(2n+\tfrac{3}{2})\ln(2n+1) - 3(2n+1) - (2n+1)\ln n + 2n - (4n+\tfrac{7}{2})\ln(4n+3) + 4n + 3.$$

The constants and terms in $n$ cancel leaving

$$\ln \Pi_n \;\sim\; (2n+3)\ln 2 \;+\; 3(2n+\tfrac{3}{2})\ln(2n+1) \;-\; (2n+1)\ln n \;-\; (4n+\tfrac{7}{2})\ln(4n+3).$$

Now make the bold step of stating that for large $n$ $\ln(2n+1) \approx \ln 2n = \ln n + \ln 2$ and $\ln(4n+3) \approx \ln 4n = \ln n + 2\ln 2$:

$$\ln \Pi_n \;\sim\; (2n+3)\ln 2 \;+\; (6n+\tfrac{9}{2})\ln n \;+\; (6n+\tfrac{9}{2})\ln 2 \;-\; (2n+1)\ln n \;-\; (4n+\tfrac{7}{2})\ln n \;-\; (8n+7)\ln 2.$$

The terms in $\ln n$ cancel leaving terms only in $\ln 2$:

$$\ln \Pi_n \;\sim\; \big(2n+3+6n+\tfrac{9}{2}-8n-7\big)\ln 2 \;=\; \tfrac{1}{2}\ln 2 \;=\; \ln\sqrt{2}.$$

This completes the proof. It might be argued that it relies too heavily on Stirling's asymptotic formula and that a more rigorous proof should be possible from Eq 1. I leave the quest to the interested reader.

## 3 Limit of product

To address the second problem posed I will follow the same approach of first expressing the limiting product in factorials then replacing the logarithm of these by Stirling's approximation. We will need

$$1\,.\,3\,.\,4\,.\,5\,.\,7\,.\,9\,.\,\ldots(2m-1) \;=\; \frac{(2m-1)!}{2^{m-1}\,(m-1)!} \;=\; \frac{(2m)!}{2^m\,m!}$$

The four sub-products are

$$2\,.\,4\,.\,6\,.\,8\,.\,\ldots\,.\,2kN \;=\; 2^{kN}(kN)!$$

$$1\,.\,3\,.\,5\,.\,\ldots\,.\,(2kN-1) \;=\; \frac{(2kN)!}{2^{kN}\,(kN)!}$$

$$1\,.\,3\,.\,5\,.\,\ldots(2k-1) \;=\; \frac{(2k)!}{2^k\,k!}$$

$$2\,.\,4\,.\,6\,.\,8\,.\,\ldots\,.\,2k \;=\; 2^k\,k!$$

Putting these together the expression is

$$\lim_{k\to\infty} 2^{2k(N-1)} \left(\frac{(kN)!}{k!}\right)^2 \frac{(2k)!}{(2kN)!}.$$

The logarithm of the argument whose limit $k \to \infty$ is sought is

$$2k(N-1)\ln 2 \;+\; 2\ln(kN)! \;-\; 2\ln k! \;+\; \ln(2k)! \;-\; \ln(2kN)!$$

When Stirling's approximation is inserted the '$-m$' terms in $(m+\frac{1}{2})\ln m - m$ cancel to leave

$$2k(N-1)\ln 2 \;+\; 2(kN+\tfrac{1}{2})\ln kN \;-\; 2(k+\tfrac{1}{2})\ln k \;+\; (2k+\tfrac{1}{2})\ln 2k \;-\; (2kN+\tfrac{1}{2})\ln 2kN.$$

Expand the logarithms:

$$2k(N-1)\ln 2 \;+\; 2(kN+\tfrac{1}{2})\ln k + \; 2(kN+\tfrac{1}{2})\ln N \;-\; 2(k+\tfrac{1}{2})\ln k$$

$$+\;(2k+\tfrac{1}{2})\ln k \;+\; (2k+\tfrac{1}{2})\ln 2 \;-\; (2kN+\tfrac{1}{2})\ln k \;-\; (2kN+\tfrac{1}{2})\ln N \;-\; (2kN+\tfrac{1}{2})\ln 2.$$

The coefficients of $\ln 2$, $\ln k$ and $\ln N$ are

$$\begin{array}{rcl} \ln 2: & 2kN - 2k + 2k + \frac{1}{2} - 2kN - \frac{1}{2} & = \; 0, \\ \ln k: & 2kN + 1 - 2k - 1 + 2k + \frac{1}{2} - 2kN - \frac{1}{2} & = \; 0, \\ \ln N: & 2kN + 1 - 2kN - \frac{1}{2} & = \; \frac{1}{2} \end{array}$$

This shows that the logarithm of the argument is $\frac{1}{2}\ln N = \ln\sqrt{N}$ consistent with the $\sqrt{N}$ in the question.

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