A continued fraction for e

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1 The question

Prove that the continued fraction

$$3+\cfrac{-1}{4+\cfrac{-2}{5+\cfrac{-3}{6+\cfrac{-4}{7+\cfrac{-5}{8+\ldots}}}}}$$

has the value $e = \exp(1)$.

This continued fraction featured in an article in the *New Scientist* magazine in July 2019. The article was headed 'Computer attempts to replicate the dream-like maths of Ramanujan' and refers to research at the Israel Institute of Technology at Haifa to use artificial intelligence to conjecture series and continued fractions which converge to e or π, following the amazing discoveries made in the early 20th century by the Indian genius Srinivasa Ramanujan. The above is a new conjecture produced by their algorithm. Their method and more results are described in an article on the internet at *www.groundai.com/project/the-ramanujan-machine-automatically-generated-conjectures-on-fundamental-constants/1*. Figure 1 is an extract from this article giving another continued fraction for e and two for π.

I wondered how one would prove that the continued fraction does converge to $\exp(1)$ and not to some close but different number The notes below are my attempt to do so.

Some years ago I wrote a monologue on continued fractions (see *www.mathstudio.co.uk*). In the compact notation used there the above would be written as $\{3 \,:\, -1, 4, -2, 5, -3, 6, -4, 7, ...\}$. The usual continued fraction for e also shows a clear pattern; it is $\{2 \,:\, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, ...\}$. The individual integers in this chain are called partial quotients and the evaluation of the continued fraction to a given depth is called a convergent. Our given continued fraction is unusual is having the minus signs. The classic way to produce continued fraction from a given ordinary fraction or decimal, as my article explains, is to use the Euclidean algorithm for the greatest common divisor of numerator and denominator. The same applies to continued fractions representing square roots. Continued fractions produced in this way have the property that their convergents each give the best approximation to the parent number amongst all fractions with about the same size of denominator.

Our MITM-RF algorithm was able to produce several novel conjectures, for example:

$$\frac{4}{\pi-2} = 3 + \cfrac{1\cdot 3}{5 + \cfrac{2\cdot 4}{6 + \cfrac{3\cdot 5}{7 + \cfrac{4\cdot 6}{8 + \cfrac{5\cdot 7}{9 + \cfrac{6\cdot 8}{10 + \cdots}}}}}}$$

$$e = 3 + \cfrac{-1}{4 + \cfrac{-2}{5 + \cfrac{-3}{6 + \cfrac{-4}{7 + \cfrac{-5}{8 + \cfrac{-6}{9 + \cdots}}}}}}$$

$$\frac{2}{\pi+2} = 0 + \cfrac{1\cdot(3-2\cdot 1)}{3 + \cfrac{2\cdot(3-2\cdot 2)}{6 + \cfrac{3\cdot(3-2\cdot 3)}{9 + \cfrac{4\cdot(3-2\cdot 4)}{12 + \cfrac{5\cdot(3-2\cdot 5)}{15 + \cdots}}}}}$$

$$e - 2 = 1 + \cfrac{2}{1 + \cfrac{-1}{1 + \cfrac{4}{1 + \cfrac{-2}{1 + \cfrac{6}{1 + \cfrac{-3}{1 + \cdots}}}}}}$$

Conjectures 1-4: Sample of automatically generated conjectures for mathematical formulas of fundamental constants, as generated by our proposed Ramanujan Machine by applying the MITM-RF algorithm. All these results are previously unknown conjectures to the best of our knowledge. Both results for π converge exponentially and both results for e converge super-exponentially. See Table 3 in the Appendix for additional results from our algorithms along with their convergence rates, which we separate to previously known formulas and new formulas.

Figure 1: Four new formulae discovered by the team at the Israel Institute of Technology.

2 Conversion to an infinite series

Many continued fractions with patterns in their partial fractions can be converted into series. My long article at *www.mathstudio.co.uk* gives several examples. The idea now is to convert the given continued factor to a series and see if we can recognise a known representation of e. In a normal continued fraction produced by the Euclidean algorithm all the partial quotients are positive and the sequence of convergents alternates about its limiting value. In our problem's continued fraction for e the minus signs produce a monotonically converging sequence.

Evaluate successive convergents. By subtracting each from the previous one, we get the terms of the series. Thus

$$3 - \frac{1}{4} = \frac{11}{4}, \qquad 3 - \frac{1}{4-\frac{2}{5}} = \frac{49}{18} \quad \text{and} \quad \frac{49}{18} - \frac{11}{4} = -\frac{1}{36}.$$

Next

$$3 - \frac{1}{4 - \frac{2}{5-\frac{3}{6}}} = \frac{87}{32} \quad \text{and} \quad \frac{87}{32} - \frac{49}{18} = -\frac{1}{288}.$$

The first 3 terms of the series are therefore $-1/4, -1/36, -1/288$. Continuing in this way, the next terms are $-1/2400, -1/21600, -1/211680$. The denominators have the general term

$$g(n) = n(n-1)\,n!\,, \qquad n \geq 2. \tag{1a}$$

The next two are g(8)=2,257,920, g(9)=26,127,360, and the continued fraction is equivalent to

$$3 - \sum_{n=2}^{\infty} \frac{1}{g(n)}\,. \tag{1b}$$

3 Summing a factorial series: an example

The exponential function can be defined as the infinite series

$$\exp x = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{so} \quad e = \sum_{n=0}^{\infty} \frac{1}{n!} \tag{2}$$

My idea was to use this as the starting place for finding a function which has the additional $n(n-1)$ in the denominators, as required for $g(n)$. Hopefully that would lead to an analytic expression proving that $\sum 1/g = 3 - e$.

My experience has been that $1/g$ has technical difficulties, so I will first illustrate the intended general method by deriving a similar infinite series which happens to have the same sum of $3 - e$. The essential point is that to introduce n and $n-1$ into the denominator of the series for $\exp x$, it seems necessary to integrate the x^n. Recall that

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} .$$

Therefore consider the function

$$xe^x = \sum_0^{\infty} \frac{x^{n+1}}{n!} .$$

The scheme is to integrate the left and right hand sides of this – the left as a function, the right as a series – and equate the two. Take the series first:

$$I_1(y) = \int_0^y \sum_0^{\infty} \frac{t^{n+1}}{n!} \, dt = \sum_0^{\infty} \left[\frac{t^{n+2}}{(n+2)n!} \right]_0^y = \sum_0^{\infty} \frac{y^{n+2}}{(n+2)n!}$$

Integrating a second time

$$I_2(x) = \int_0^x I_1(y) \, dy = \sum_0^{\infty} \frac{x^{n+3}}{(n+3)(n+2)n!} .$$

At $x = 1$

$$I_2(1) = \sum_0^{\infty} \frac{1}{(n+3)(n+2)n!} .$$

Now the left hand side. Integrating by parts with $u = t$, $e^t dt = dv$ gives

$$\int_0^y te^t \, dt = [te^t - e^t]_0^y = (y-1)e^y + 1 = J_1(y).$$

Integrating by parts a second time

$$\int_0^x J_1(y) \, dy = [(y-2)e^y + y]_0^x = (x-2)e^x + x + 2$$

and when $x = 1$ this has the value $3 - e$. The conclusion is that

$$\sum_0^{\infty} \frac{1}{(n+3)(n+2)n!} = \sum_0^{\infty} \frac{n+1}{(n+3)!} = 3 - e . \tag{3}$$

Some numerical values of successive partial sums will compare the convergence of this series with $\sum_2 1/g(n)$ in Eq 1. The first six partial sums of each are as follows:

$$\text{Eq 1a,b}: \quad 0\cdot 250, \quad 0\cdot 2778, \quad 0\cdot 28125, \quad 0\cdot 28167, \quad 0.281713, \quad 0\cdot 281718.$$

$$\text{Eq 3}: \quad 0\cdot 167, \quad 0\cdot 250, \quad 0\cdot 2750, \quad 0\cdot 28055, \quad 0\cdot 281548, \quad 0\cdot 281696,$$

A precise value is $3 - e = 0\cdot 281718171541$.. Both Eqs 1 and 3 achieve this accuracy with 13 terms, though Eq 1 initially converges faster. The Euclidean algorithm produces the continued fraction $3 - e = \{0 \,:\, 3, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots\}$ and its convergents are

$$\tfrac{1}{3}, \quad \tfrac{1}{4}, \quad \tfrac{2}{7}, \quad \tfrac{9}{32} = 0\cdot 28125, \quad \tfrac{11}{39} = 0\cdot 28205, \quad \tfrac{20}{71} = 0\cdot 28169, \quad \tfrac{131}{465} = 0\cdot 281720, \quad \ldots.$$

4 Linking e to $\sum 1/g(n)$

If we try to apply the method of §2 to $\sum_2 1/g(n)$, the obvious function to choose is

$$\frac{e^x}{x^2} = \sum_{n=0}^{\infty} \frac{x^{n-2}}{n!} \tag{4}$$

since integrating this twice will bring down $n-1$ then n into the denominator. Unfortunately e^x/x^2 is much less well behaved than xe^x. It has a singularity at $x = 0$ so the lower limit of integration cannot be set to 0, but instead to some small value $a > 0$ and at some point the limit $a \to 0$ taken. In addition there is no closed form for its integrals. Instead it is represented by a special function $\mathrm{Ei}(x)$ which in turn is expressed by an infinite series. That clearly brings the risk that evaluating the two sides of Eq 4 will lead only to a tautology. So we venture forth with caution and fingers crossed.

4.1 Right side

The series is

$$I_0(t) = \frac{1}{t^2} + \frac{1}{t} + \sum_{n=2}^{\infty} \frac{t^{n-2}}{n!}$$

and the first integral is from a to y:

$$I_1(y) = \int_a^y I_0(t)\,dt = \left[-\frac{1}{t} + \ln t + \sum_2^{\infty} \frac{t^{n-1}}{(n-1)n!}\right]_a^y$$

The second integral is from a to 1

$$I_2(y) = \int_a^1 \left[-\frac{1}{y} + \ln y + \sum_2^{\infty} \frac{y^{n-1}}{(n-1)n!} + \frac{1}{a} - \ln a - \sum_2^{\infty} \frac{a^{n-1}}{(n-1)n!}\right] dy$$

$$= \left[-\ln y + y\ln y - y + \sum_2^{\infty} \frac{y^n}{n(n-1)n!}\right]_a^1 + y\left[\frac{1}{a} - \ln a - \sum_2^{\infty} \frac{a^{n-1}}{(n-1)n!}\right]_a^1.$$

With $\ln 1 = 0$ this evaluates to

$$-1 + \sum_2 \frac{1}{g(n)} + \ln a - a\ln a + a - \sum_2^{\infty} \frac{a^n}{n(n-1)n!} + \frac{1}{a} - \ln a - \sum_2^{\infty} \frac{a^{n-1}}{(n-1)n!} - 1 + a\ln a + \sum_2^{\infty} \frac{a^n}{(n-1)n!},$$

$$I_2(1) = \sum_2 \frac{1-a^n}{g(n)} - 2 + a + \frac{1}{a} - (1-a)\sum_2^{\infty} \frac{a^{n-1}}{(n-1)n!}. \tag{5}$$

I carried out a few numerical checks to satisfy myself that this is correct. Clearly the limit $a \to 0$ cannot be taken yet because of the $1/a$ term which dominates for small a. It is to be hoped that it will cancel when the left side is evaluated.

4.2 Left side

Here we meet the exponential integral Ei(x), a special function tabulated in the handbook by Abramowitz and Stegun and on-line at the NIST Digital Library of Mathematical Functions. It is defined as the principal value of

$$\mathrm{Ei}(x) \;=\; \int_{-\infty}^{x} \frac{e^x}{x}\,dx, \qquad x > 0$$

where the path of integration avoids the origin.

Start by integrating e^x/x by parts in two ways.

Method A Let $u = 1/x$, $dv = e^x\,dx$ to obtain

$$\int \frac{e^x}{x}\,dx \;=\; \frac{e^x}{x} + \int \frac{e^x}{x^2}\,dx \;+\; C$$

where C is some constant. From this

$$J_1(y) \;=\; \int_a^y \frac{e^t}{t^2}\,dt \;=\; \left[-\frac{e^t}{t}\right]_a^y + \int_a^y \frac{e^t}{t}\,dt\,.$$

This generates our required first integral $J_1(y)$ as

$$J_1(y) \;=\; -\frac{e^y}{y} + \frac{e^a}{a} + \mathrm{Ei}(y) - \mathrm{Ei}(a)\,. \tag{6}$$

Method B Let $u = e^x$, $dv = dx/x$ to obtain

$$\mathrm{Ei}(x) \;=\; e^x \ln x - \int e^x \ln x\,dx.$$

The integral on the right can only be performed by appeal to the series expansion of e^x. Again by parts

$$\int e^x \ln x\,dx. \;=\; \sum_0^{\infty} \int \frac{x^n \ln x}{n!}\,dx \;=\; \sum_0^{\infty} \frac{x^{n+1}}{(n+1)!}\left(\ln x - \frac{1}{n+1}\right)$$
$$=\; (e^x - 1)\ln x - \sum_1^{\infty} \frac{x^n}{n\,n!}\,.$$

Putting this result into the above expression for Ei(x) gives

$$\mathrm{Ei}(x) \;=\; \ln x + \sum_1^{\infty} \frac{x^n}{n\,n!} + C \tag{7}$$

where C is a constant. Abramowitz and Stegun give this result in their book at §5.1.10 with $C = \gamma$, Euler's constant whose value is $0{\cdot}5772$.... However, we do not have to be concerned with this since it cancels when integration is between finite limits. They do not give $\int \mathrm{Ei}(x)$ but it can be found from Eq 7 as follows:

$$\int \mathrm{Ei}(x)\,dx \;=\; x\ln x - x + \sum_1^{\infty} \frac{x^{n+1}}{(n+1)n\,n!}\,.$$

Split $1/(n(n+1))$ into partial fractions as $1/n - 1/(n+1)$ to obtain

$$\int \mathrm{Ei}(x)\,dx \;=\; x\ln x + \sum_1^{\infty} \frac{x^{n+1}}{n\,n!} \;\; - x - \sum_1^{\infty} \frac{x^{n+1}}{(n+1)n!}$$

$$= \; x\text{Ei}(x) - x - \sum_2^\infty \frac{x^n}{n!} \; = \; x\text{Ei}(x) - x - (e^x - 1 - x) \; = \; x\text{Ei}(x) - e^x + 1\,. \tag{8}$$

This places us in a position to carry out the second integration of e^x/x^2. From Eq 6

$$\int_a^1 J_1(y)\,dy \; = \; \left[-\text{Ei}(y) + y + y\frac{e^a}{a} + y\text{Ei}(y) - e^y - y\text{Ei}(a)\right]_a^1 .$$

$$= \; \frac{e^a}{a} - e\,. \tag{9}$$

4.3 Matching Left with Right

With all the integrations carried out, the left side of Eq 4 can be equated to the right: Eq 5 to Eq 9.

$$\frac{e^a}{a} - e \; = \; \sum_2 \frac{1-a^n}{g(n)} - 2 + a + \frac{1}{a} - (1-a)\sum_2^\infty \frac{a^{n-1}}{(n-1)n!}\,.$$

and the limit $a \to 0$ taken. Near this limit $e^a/a \approx 1/a + 1$ and the $1/a$ cancel across the two sides as we had hoped. The above two sides approach

$$1 - e \; = \; \sum_2 \frac{1}{g(n)} - 2$$

proving that

$$\sum_2 \frac{1}{g(n)} \; = \; 3 - e.$$

This coincides with Eq 1b and so confirms that the continued fraction found by the Israel Institute of Technology team does indeed equal e.