Q16 : Consider the power series expansion

$$\frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer $n \geq 0$, there is an integer m such that

$$a_n^2 + a_{n+1}^2 = a_m.$$

This was problem A-3 in the 1999 Putnam mathematical competition.

The obvious first step seems to be to find the coefficients a_n using the binomial theorem. We are not told that this series converges, so it is to be regarded as a formal power series only – a vehicle for generating the series of coefficients. The binomial expansion of $(1+u)^n$ is

$$1 + nu + \frac{n(n-1)}{2!}u^2 + \frac{n(n-1)(n-2)}{3!}u^3 + \frac{n(n-1)(n-2)(n-3)}{4!}u^4 + \ldots.$$

Taking $u = -x(2+x)$ and $n = -1$,

$$\frac{1}{1-2x-x^2} = 1 + x(2+x)^1 + x^2(2+x)^2 + x^3(2+x)^3 + x^4(2+x)^4 + \ldots$$

$$= 1 + (2x+x^2) \; + \; x^2(4+4x+x^2) \; + \; x^3(8+12x+6x^2+x^3) \; + \ldots$$

$$= 1 + 2x + 5x^2 + 12x^3 + \cdots \; . \tag{1}$$

For the higher powers of $(2+x)^n$ we can invoke the binonial theorem again :

$$(2+x)^n \; = \; {}^nC_0\, 2^n x^0 + \; {}^nC_1\, 2^{n-1}x^1 + \; {}^nC_2\, 2^{n-2}x^2 + \; \ldots {}^n C_{n-1}\, 2^1 x^{n-1} + \; {}^nC_n\, 2^0 x^n$$

where ${}^nC_r$ is the 'from n choose r' binomial coefficient. Using this, the sequence of coefficents is found to begin

$a_0 = 1, \; a_1 = 2, \; a_2 = 5, \; a_3 = 12, \; a_4 = 29, \; a_5 = 70, \; a_6 = 196, \; a_7 = 408, \; a_8 = 985, \; a_9 = 2378.$

Each of these a_n is a sum of powers of 2 and binomial coefficents, as explored later. These integers are called Pell numbers.

We have to find m so that $a_n^2 + a_{n+1}^2 = a_m$. Starting with $n = 0$, $1^2 + 2^2 = 1 + 4 = 5$, and this is the value of a_2. The next pair gives $2^2 + 5^2 = 4 + 25 = 29$ which is a_4. The next pair is $5^2 + 12^2 = 25 + 144 = 169 = a_6$. So it looks as if the pattern is

$$a_n^2 + a_{n+1}^2 = a_{2n+2} \; ; \qquad \text{that is} \;\; m = 2n+2. \tag{2}$$

As a check $a_4^2 + a_5^2 = 29^2 + 70^2 = 841 + 4900 = 5741$, which should be a_{10}. By direct computation of the terms in x^{10}, a_{10} works out to be

$$a_{10} = {}^{10}C_0\, 2^{10} \; + \; {}^9C_1\, 2^8 + \; {}^8C_2\, 2^6 + \; {}^7C_3\, 2^4 + \; {}^6C_4\, 2^2 + \; {}^5C_5\, 2^0 = 5741,$$

supporting the conjecture, Eq 2.

How to prove this conjecture ? It will help to note a simpler, linear recursion relation between the coefficients :

$$a_{n+1} = 2a_n + a_{n-1}. \qquad (3)$$

For instance, $a_2 = 5 = 2.2 + 1$, $a_3 = 12 = 2.5 + 2$, $a_4 = 29 = 2.12 + 5$, $a_{10} = 2.2378 + 985$. At this stage Eq 3 is not yet proven.

Another interesting pattern is quite easy to spot. Consider Table 1 which lists a_j and a_k and their product $a_j a_k$. The table is symmetrical about its diagonal, so only the upper right half is shown. Observe how the sum of any adjacent diagonal pair, upper left + lower right, gives another Pell number. For instance, $10 + 60 = 70 = a_5$, $58 + 350 = 408 = a_7$. Figure 1 is an annotated copy of Table 1 with lines tieing related numbers. The relationship is

$$a_j\, a_k \;+\; a_{j+1}\, a_{k+1} \;=\; a_{j+k+2}. \qquad (4)$$

Clearly, with $j = k$ we recover the conjectured relation Eq 2, so this is just a special case in Table 1, corresponding to pairs of values down the main diagonal.

	k	0	1	2	3	4	5	6	7
	a_k	1	2	5	12	29	70	169	408
j	a_j								
0	1	1	2	5	12	29	70	169	408
1	2		4	10	24	58	140	338	816
2	5			25	60	145	350	845	2040
3	12				144	348	840	2028	4896
4	29					841	2030	4901	11832
5	70						4900	11830	28560
6	169							28561	68952
7	408								166464
8	985								

TABLE 1 – Products $a_j a_k$. Note sum of each diagonal pair

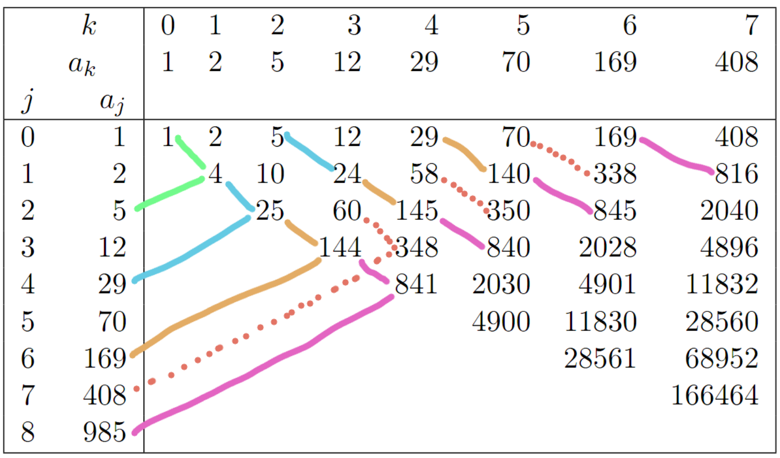

FIGURE 1 – Table 1 annotated to show relations between numbers.

From the linear recursion relation Eq 3 an exact expression for each a_n can be found. Let $\mathcal{D}$ denote the operation of increasing the index n by 1. Eq 3 is

$$\mathcal{D}(\mathcal{D}(a_n)) \;=\; 2\mathcal{D}(a_n) + a_n \qquad \text{or} \qquad [\mathcal{D}^2 - 2\mathcal{D} - 1]a_n \;=\; 0.$$

Treating the operator $\mathcal{D}$ as a quasi-variable, this quadratic has solutions $\mathcal{D}a_n = \beta_{1,2}\, a_n$ or

$$a_{n+1} = \beta_1 a_n, \quad a_{n+1} = \beta_2 a_n \quad \text{where} \;\; \beta_1 = 1 + \sqrt{2}, \;\; \beta_1 = 1 - \sqrt{2}. \tag{5}$$

Each of these expresses a geometric series with common ratio β_1 or β_2. Note that $\beta_1\beta_2 = -1$. The most general solution is a sum of these two series with arbitrary weightings K_1, K_2 : that is, $K_1\beta_1^n + K_2\beta_2^n = a_n$. In our case we need to choose K_1, K_2 to fit the coefficients $a_0 = 1$, $a_1 = 2$, etc. We therefore want :

$$n = 0: \quad K_1 + K_2 = 1, \qquad\qquad n = 1: \quad K_1\beta_1 + K_2\beta_2 = 2.$$

The simultaneous solutions are

$$K_1 \;=\; \frac{\sqrt{2}\,+1}{2\sqrt{2}}, \qquad K_2 \;=\; \frac{\sqrt{2}\,-1}{2\sqrt{2}}. \tag{6}$$

One can then check that $K_1\beta_1^2 + K_2\beta_2^2 = 5 = a_2$, $\;K_1\beta_1^3 + K_2\beta_2^3 = 12 = a_3$, etc.

Now substitute these values into the left and right sides of Eq 4, which is yet to be verified. The right hand side is

$$a_{j+k+2} = K_1\beta_1^{j+k+2} + K_2\beta_2^{j+k+2},$$

whilst on the left side are

$$a_j a_k = K_1^2\beta_1^{j+k} + K_1K_2(\beta_1^j\beta_2^k + \beta_1^k\beta_2^j) + K_2^2\beta_2^{j+k}, \tag{7a}$$

$$a_{j+1}a_{k+1} = K_1^2\beta_1^{j+k+2} + K_1K_2(\beta_1^{j+1}\beta_2^{k+1} + \beta_1^{k+1}\beta_2^{j+1}) + K_2^2\beta_2^{j+k+2}. \tag{7b}$$

This looks quite complicated, but the terms in K_1K_2, when added, cancel because $\beta_1\beta_2 = -1$:

$$K_1K_2(\beta_1^j\beta_2^k + \beta_1^k\beta_2^j)(1 + \beta_1\beta_2) = 0.$$

This leaves

$$a_j a_k + a_{j+1}a_{k+1} = K_1^2\beta_1^{j+k}(1 + \beta_1^2) + K_2^2\beta_2^{j+k}(1 + \beta_2^2).$$

If we can now show that $K_1(1 + \beta_1^2) = \beta_1^2$, and similarly for K_2, Eq 4 will be proved.

$$\beta_1^2 = 3 + 2\sqrt{2} \qquad \text{so} \qquad K_1(1 + \beta_1^2) = \frac{(\sqrt{2}\,+1)}{2\sqrt{2}}(4 + 2\sqrt{2}) = 3 + 2\sqrt{2}.$$

$$\beta_2^2 = 3 - 2\sqrt{2} \qquad \text{so} \qquad K_2(1 + \beta_2^2) = \frac{(\sqrt{2}\,-1)}{2\sqrt{2}}(4 - 2\sqrt{2}) = 3 - 2\sqrt{2}.$$

Thus Eq 4 is established, and with it the special case posed in the original Putnam question.

To be rigorous, the proof does rest on the linear recursion relation in Eq 3 also being correct, so strictly this should be verified. Here is one means of proving it, based on enumerating the terms in the expansion of $x^n(2+x)^n$ which give rise to the coefficients a_n. The notation and numbers of terms become rather opaque as n increases, so here first is a simpler example : that $a_5 = 2a_4 + a_3$. Start by expanding $x^n(2+x)^n$:

$$x^2(2+x)^2 = x^2({}^2C_0\,2^2x^0 + {}^2C_1\,2^1x^1 + {}^2C_2\,2^0x^2)$$

$$x^3(2+x)^3 = x^3({}^3C_0\,2^3x^0 + {}^3C_1\,2^2x^1 + {}^3C_2\,2^1x^2 + {}^3C_3\,2^0x^3)$$

$$x^4(2+x)^4 = x^4({}^4C_0\,2^4x^0 + {}^4C_1\,2^3x^1 + {}^4C_2\,2^2x^2 + {}^4C_3\,2^1x^3 + {}^4C_4\,2^0x^4)$$

$$x^5(2+x)^5 = x^5({}^5C_0\,2^5x^0 + {}^5C_1\,2^4x^1 + {}^5C_2\,2^3x^2 + {}^5C_3\,2^2x^3 + {}^5C_4\,2^1x^4 + {}^5C_5\,2^0x^5).$$

Picking out the terms in x^3, x^4 and x^5 respectively

$$a_3 = {}^3C_0\,2^3 + {}^2C_1\,2^1.$$

$$2a_4 = {}^4C_0\,2^5 + {}^3C_1\,2^3 + {}^2C_2\,2^1,$$

$$a_5 = {}^5C_0\,2^5 + {}^4C_1\,2^3 + {}^3C_2\,2^1.$$

The terms in 2^5 are equal because ${}^nC_0 = 1$ for all n. For terms in 2^3 and 2^1 we wish to show that

$${}^3C_0 + {}^3C_1 = {}^4C_1 \quad \text{and} \quad {}^2C_1 + {}^2C_2 = {}^3C_2.$$

But these are true – they express the famous summation relation in Pascal's triangle. This method can be generalised using

$$a_{2k} = \sum_{j=0}^{k} {}^{2k-j}C_j\,2^{2(k-j)} \quad \text{and} \quad a_{2k+1} = \sum_{j=0}^{k} {}^{2k-j+1}C_j\,2^{2(k-j)+1},$$

but I will not labour this any further. The answer has been thoroughly confirmed.

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