Q15 : Prove that there are unique integers a, n such that

$$a^{n+1} - (a+1)^n = 2001.$$

This was problem A-5 in the 2001 William Lowell Putnam mathematical competition – hence the number 2001 featuring.

We are asked to find two pairs of consecutive positive integers a, $a+1$ and n, $n+1$, and show that they are unique. We could just use a pocket calculator to hunt for a solution, but some analysis and intuition will shorten the search. Entering a few values of a and n into a calculator quickly shows that n is likely to be small, because otherwise $a^{n+1} - (a+1)^n$ is large and varies wildly. For example

$$5^9 - 6^8 = 273509 \quad \text{but} \quad 5^{10} - 6^9 = -312071.$$

So let's start with $n = 1$ and work upwards.

$n = 1$ gives the quadratic $a^2 - a - 2002 = 0$. This has discriminant 8009 which is prime so there are no solutions. Moving up to $n = 2$ we have the cubic

$$a^3 - a^2 - 2a - 2002 = 0.$$

Any integer root must be a factor of $2002 = 2.7.11.13$. 13 is quickly found to be a root : $13^3 - 14^2 = 2197 - 196 = 2001$. So at least one solution is $a = 13$, $n = 2$, and this is the only solution for $n = 2$.

Uniqueness? – can we show that there are no solutions for $n \geq 3$? Using the binomial expansion of $(a+1)^n$, the equation to solve has the form $a^{n+1} - a^n - na^{n-1} - \cdots - na = 2002$. Clearly, any integer root a_0 must be a factor of 2002, and $(a - a_0)$ a linear factor of the polynomial $P_n = a^{n+1} - a^n - na^{n-1} - \cdots - na - 2002$. The complete list of factors (and I include the negative ones) is

$$\pm(1, 2, 7, 11, 13, 14, 22, 26, 77, 91, 143, 154, 182, 286, 1001, 2002).$$

However several of these can be eliminated :

1. $a \neq \pm 1$ since no power of 2 satisfies $1 - 2^n = 2001$.
2. $a \neq \pm 2002$. The + case would require that $2002^{n+1} - 2003^n = 2001$, equivalent to

$$2002 = \left(1 + \frac{1}{2002}\right)^n + \left(\frac{2001}{2002^n}\right).$$

   For low n the right hand side is tiny; even for $n = 1000$, $(1 + \frac{1}{2002})^n$ is only $1{\cdot}65$, and $2001/2002^n$ is practically zero. Taking logs, we find that the value of n which makes $a^{n+1} = (a+1)^n$ is

$$n_c = \frac{\ln a}{\ln\left(\frac{a+1}{a}\right)}.$$

   For $a = 2002$ this is $15222{\cdot}81$, an enormous exponent and one which is insufficiently close to an integer to indicate a possible solution. (This line of thought is applied to other factors of 2002 below.)

3. 2001 = 3.23.29 so we can readily gain information by listing all the possibilities mod 3 (see Table 1). The only combination which satisfies $a^{n+1} - (a+1)^n \equiv 0 \bmod 3$ is $a \equiv 1 \bmod 3$, $n \equiv 0 \bmod 2$. Naturally our solution $a = 13$, $n = 2$ satisfies these.

a	a+1	a^{n+1}	(a+1)^n	difference
0	1	0	1	2
1	2	1	2 (n odd), 1 (n even)	2 (n odd), 0 (n even)
2	0	1 (n odd), 2 (n even)	0	1 (n odd), 2 (n even)

TABLE 1 – Values of $a^{n+1} - (a+1)^n \bmod 3$

These considerations reduce the positive and negative candidate factors, a, to

$$7, 13, 22, 91, 154, 286,$$

$$-2, -11, -14, -26, -77, -143, -182, -1001.$$

and constrains n to be even. For any chosen n we can use trial substitution of these factors of 2002. For instance, $P_4 = a^5 - a^4 - 4a^3 - 6a^2 - 4a - 2002$, and none of the above factors is a zero, so there are no solutions for $n = 4$.

More can be deduced from the observation above (item 2) that $a^{n+1} > (a+1)^n$ only if $n < n_c = \ln(a)/\ln((a+1)/a)$. This sets a ceiling on possible values of n. For instance, for $a = 7$, $n_c = 14{\cdot}57$, and for $a = 13$, $n_c = 34{\cdot}61$. Table 2 below lists the values of $a^{n+1} - (a+1)^n$ for both $a = 7$ and 13. Values increase rapidly until a maximum n_{max} (see Appendix) then when n_c is passed there is a change of sign from + to −. Unless the target integer, 2001, is found for small n (as it is for $a = 13$, $n = 2$), the only other possibility is that n_c is so very close to an even integer that $a^{n+1} - (a+1)^n$ is relatively very close to zero – and equal to 2001. So in practice we need to check only a few values of a and n. Then no other solutions are found.

Finally a comment about irreducible polynomials. Finding a linear factor is closely related to determining whether P_n is irreducible over the integers. One well known test is Eisenstein's, but it is quite restrictive because it requires that all coefficients other than the leading one be divisible by the same prime. Sometimes the criterion can apply only after a shift of variable from a to $a + k$ for some experimentally determined integer k. However, I have been unable to find a shift k which transforms any of the P_n, for $n = 6$, 8 or 10, into a form where one prime divides all the required coefficients. For example, for $n = 6$

$$P_6(a) = a^7 - a^6 - 6a^5 - 15a^4 - 20a^3 - 15a^2 - 6a - 2002,$$

$$P_6(a+1) = a^7 + 6a^6 + 9a^5 - 25a^4 - 125a^3 - 219a^2 - 185a - 2064.$$

Here each of the coefficients 6, 9, 219 and 2064 is divisible by 3 but not by 3^2 (good!), but none of 25, 125 and 185 has 3 as factor, so Eisenstein's test is not applicable here.

I rest my case on having found the solution $n = 2$, $a = 13$, and having proved that any other solution must come from the finite number of possibilities prescribed by

1. a one of a short list of factors of 2002, namely 7, 13, 22, 91, 154, 286.
2. n even

n	$7^{n+1}-8^n$	$13^{n+1}-14^n$
1	41	155
2	279	2001
3	1889	25817
4	12711	332877
5	84881	4288985
6	561399	55218981
7	3667649	710317217
8	23576391	9128710317
9	148257521	1.17197E+11
10	903584919	1.50291E+12
11	5251352609	1.92485E+13
12	28169533671	2.46181E+14
13	1.28467E+11	3.14366E+15
14	3.49515E+11	4.00739E+16
15	−1.95144E+12	5.09849E+17
16	−4.88445E+13	6.47246E+18
33	−5.79708E+29	8.42097E+36
34	−4.69178E+30	4.30638E+37
35	−3.79131E+31	−3.6989E+38
36	−3.05956E+32	−1.7825E+40

TABLE 2 – $a^{n+1} - a^n$ for $a = 7$ and 13.

3. $n \geq 6$
4. $n < n_c = \ln(a)/\ln((a+1)/a)$ and hence in all circumstances $n \leq 1620$, corresponding to $a = 286$.

This reduces the extent of any search to about a dozen values, and the only solution thereby found is $n = 2$, $a = 13$.

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1 Appendix

Figure 1 shows the increase in $a^{n+1} - (a+1)^n$ for $a = 7$. Starting from $n = 1$, the pattern of increase with n for any fixed a, for $n < n_c$, can be proved as follows. Suppose that

$$a^{n+1} - (a+1)^n = k > 0$$

Multiplying by a, $\quad a^{n+2} - a(a+1)^n = a^{n+2} - (a+1)^{n+1} + (a+1)^n = ak.$

Now $\quad a^{n+2} - (a+1)^{n+1} > k \;$ if $\; ak - (a+1)^n > k, \;$ or $\; k > \dfrac{(a+1)^n}{a-1} > (a+1)^{n-1}.$

For $n = 1$ this requires only that $k > 1$, and this condition is met by a wide margin for all listed factors of 2002.

The increase continues to the maximum in the curve at n_{max} where

$$n_{max} = \frac{\ln a + \ln\ln a - \ln\ln(a+1)}{\ln(a+1) - \ln a}.$$

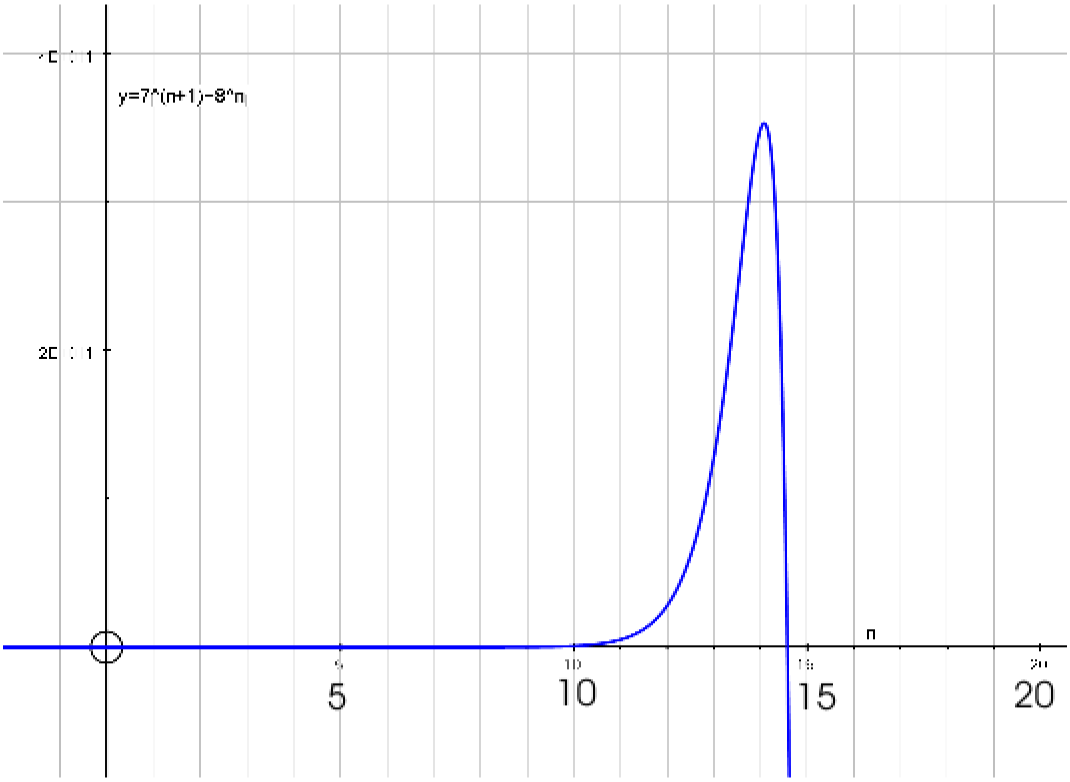

FIGURE 1 – Graph of $7^{n+1} - 8^n$ versus n.

For $a = 7$ $n_{max} = 14{\cdot}08$ and for $a = 13$ $n_{max} = 34{\cdot}23$. Both these values are very close to the respective values of n_c ($14{\cdot}57$ and $34{\cdot}61$) where the curve, crosses the abscissa.