Moving a large sofa round a tight corner

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1 The problem

According to Wikipedia this idealisation of a common practical problem of daily life has not been fully solved. A sofa or similar object has to be moved from one room to another. The two rooms are connected by an L-shaped passageway of unit width with a right-angled turn. What is the shape and size of the largest sofa which can be moved without becoming stuck? In the mathematical idealisation the 'sofa' has uniform cross section so the problem is reduced to two dimensions, and the sofa is not necessarily of a shape that anyone could sit on comfortably.

According to Wikipedia the problem was stated in mathematical terms in 1966 by the Canadian Leo Moser, but had been around as a brain teaser long before. The so-called 'sofa constant' S is the cross-sectional area of the largest object which can be moved, and the challenge is threefold: i) to find its exact value, ii) to prove that it is indeed the largest, and iii) to demonstrate a sofa which has this area. Several mathematicians have applied themselves to this optimisation problem and S is believed to lie between the bounds $2 \cdot 2195$ and $2 \cdot 37$. Figure 1 shows a sofa designed by Scottish mathematician John Hammersley with area $\pi/2 + 2/\pi = 2 \cdot 2074$, which must be close to the limit. Hammersley made contributions to optimization problems such as the travelling salesman problem. For this article I have not looked at the literature, but simply record my own amateur exploration of the matter. My search has not been as successful as Hammersley's, though I do describe a simple constructive

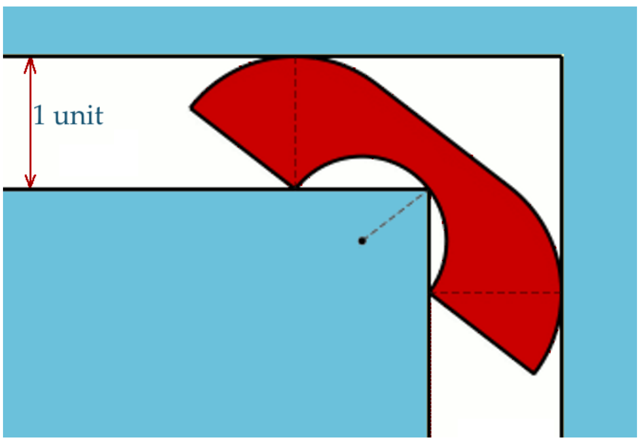

Figure 1: Moving a large 2-D sofa down a corridor and round a tight corner. (*Diagram of sofa proposed by Hammersley, from Wikipedia.*)

algorithm which, after some iteration, demonstrates a 'sofa' with area $2 \cdot 194$ square units. Two video clips illustrating two shapes of sofa moving through the corridor accompany this article on *www.mathstudio.co.uk*.

2 Some simple shapes

Since the corridor is 1 unit wide, a square with side 1 unit is the largest object which can be moved without turning. Neglecting friction with the walls, it can be slid west to east along the first corridor into the corner then north to south down the other one, giving a lower bound for S of 1. Larger objects must be turned, and clearly a semicircle of radius 1 will turn as Figure 2 illustrates. This places $S > \pi/2 = 1 \cdot 57$. The figure makes clear that

1. the largest object will have mirror symmetry about its centre line, imposed by both corridors being the same width, and
2. if the object has a hollow in the middle of its lower surface, the sharp inner corner of the walls can fit into this concavity, allowing the object to be larger in other parts. Hammersley's sofa has this form and so do all other proposed shapes.
3. we might expect that one critical position is when the object has turned through 45°, though there may be shapes for which other angles are also critical.

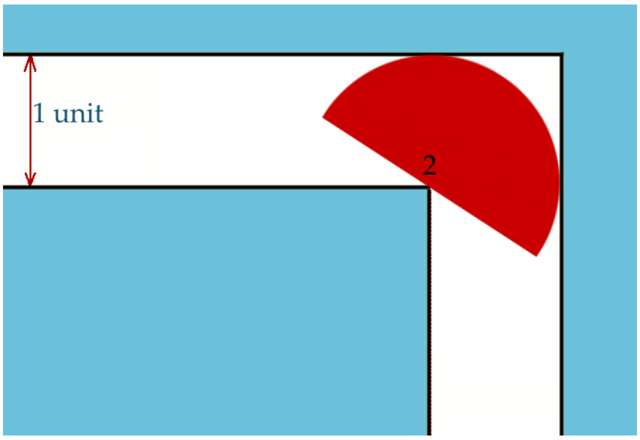

Figure 2: A semicicle of radius 1 will turn.

Before examining sofa-like objects, I divert briefly to consider straight and curved sticks (curved surfaces seen edge-on). Assuming a 2-D problem, the longest straight stick of negligible thickness which can be manoeuvred has length $2\sqrt{2} \approx 2 \cdot 83$, this being the distance across the corner when the stick is at 45° to the walls. Suppose now that the stick is curved into an arc of a circle radius r which subtends angle 2β radians at its centre of curvature, O, so its arc length, s, is $2r\beta$. The geometry is shown in Figure 3. The distance CP from the centre C of the stick to either tip T_1, T_2, measured parallel to the radius through C, is $r(1 - \cos\beta)$ and this cannot exceed 1, the width of the corridor. Given a value for β, the

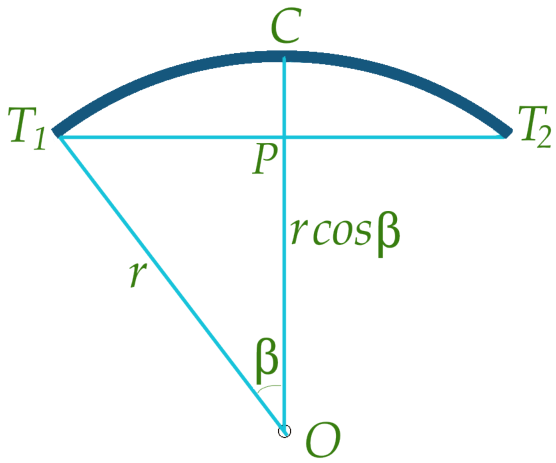

Figure 3: Object in the form of a circular arc of negligible thickness.

largest radius for which the arc will still fit into either corridor is

$$r = \frac{1}{1-\cos\beta} \quad \text{and the arc length} \quad s = \frac{2\beta}{1-\cos\beta}.$$

If we picture an arc of fixed length s turning the corner, it clearly needs to bend, and the case $\beta = \pi/4$, the quarter circle, is significant because when the arc is half way round the corner and C is at the sharp inner corner, the tangents to the arc at the tips T_1, T_2 are coincident with the outer corridor walls, as the left panel of Figure 4 illustrates. The radius at $\beta = \pi/4$ is $2+\sqrt{2} \approx 3{\cdot}414$. With this radius fixed, the arc could be extended to larger β and still fit at the corner, but it would then be too large to fit into the corridor. Therefore $\beta = \pi/4$, $r = 2+\sqrt{2}$ is the largest arc which can pass through the corridor and still turn the corner. Referring to Figure 4 this quarter circle can pass because two conditions coincide:

- the distance CP is $r(1-\frac{1}{\sqrt{2}}) = 1$, requiring $r = 2+\sqrt{2}$,
- the perpendicular distance of C at the sharp corner from either outer wall is 1, and also equal to $r - r/\sqrt{2}$, making $r = \sqrt{2}/(\sqrt{2}-1) = 2+\sqrt{2} \approx 3\cdot 414$ again.

The length of this optimum arc is $\pi(2+\sqrt{2})/2 \approx 5\cdot 363$, about 90% longer than the longest straight rod. The distance $PT_1 = PT_2 = 1+\sqrt{2} \approx 2\cdot 41$.

This 1-D case could in principle be a base for designing 2D objects through giving the quarter-circle arc some thickness. For instance, the right panel of Figure 4 shows how a 'foot' could be created at T_1 by building along the line T_1P for a distance no greater than $\sqrt{2}$ and filling the inner edge between this and the foot at T_2 by a concave notch, perhaps a semi-ellipse or even a unit semicircle as drawn. However, having looked briefly at analytical approaches, I have concluded that there are too many possibilities of shape and size, and the perimeter would probably be made of piecewise curves whose algebra could be complicated and tedious. For example, even a simple segment of an annulus would pose the choice of how the ends were shaped. They could be straight radial cuts from outer circle to inner, or straight oblique cuts, or rounded. In view of this I have followed an intuitive 'engineering' approach as described in the next section.

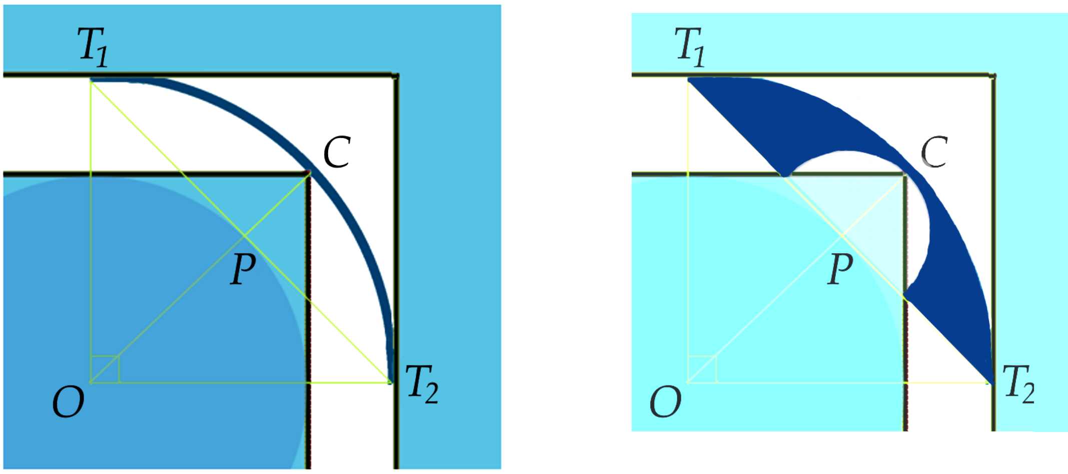

Figure 4: Left: a quarter-circle arc turning the corner. Right: a 2-D object made by extending a foot from each end.

3 A computational approach

The construction method I propose is to fill the straight section of corridor with sofa material and shave slices off it as it turns so that at each angle it fits within the space. Since as the angle of rotation increases either something or nothing will be pared away, its area will decrease monotonically up to 45° and so will fit the corridor at any angle. In other words we start with an object which fills one of the corridors, move it into the corner, then turn it in small increments, deleting all elements which move outside the lines defining the two corridors. At each angle of rotation we

- impose mirror symmetry on the object about its mid-line, and
- adjust the horizontal and vertical positions relative to the walls to maximise the retained area.

The algorithm is as follows:

1. define an origin and scale. For convenience the object will stay still with its centre base at $(0,0)$ while slanting lines representing the walls will be placed over the object at each angle θ.

2. represent the right half of the object by nodes on the square grid at a suitably high density with horizontal and vertical node spacings $dx = dy$. Make the object symmetric about the line $x = 0$. I used a simple rectangle as the starting shape, 1 unit high in y to fill the corridor, and half length d in x.

3. place over this rectangle a corridor with corner at angle θ up to 45°, and adjust the corridor's position by varying the position (a, b) of the sharp inner corner. At each angle and position (a, b) calculate the area within the walls and so determine (a, b) for largest area. Delete the nodes and cells outside of the lines representing the corridors.

4. step the angle of turn from 0 to 45°, deleting all nodes which lie outside the corridors. The final shape is guaranteed to pass round the corner and should approach in area the largest possible sofa subject for the starting half-length d.

I have written a computer program to carry out this algorithm. The variables are

- d, the half length of the starting rectangle.
- the spacing of the nodes in the grid, dx, dy. I used $0 \cdot 05$ for quick calculations and $0 \cdot 02$, $0 \cdot 01$ and even $0 \cdot 005$ for detailed studies at selected values of d.
- the increment in angle θ. I used 1 degree for quick calculations and half degree for final ones.
- a length ds used in the procedure for searching for the optimum (a, b) at any θ. This was usually equal to dx.

Some details of the program are

1. The grid is n_x nodes by n_y, with the nodes indexed 0 to $n_x n_y - 1$. The array element at each index stores the x and y co-ordinates plus a flag F which is set to 0 if the node lies inside or exactly on one of the corridor walls, as assessed up to the current value of θ, and to a different value if outside.
2. The inner sharp corner is at (a, b) and the outer corner at $(\cos\theta - \sin\theta + a, \cos\theta + \sin\theta + b)$. The walls at rotation θ are given by these straight lines up to these corners:
$$\text{upper west-east wall:} \quad y - (\cos\theta + \sin\theta + b) = [x - (\cos\theta - \sin\theta + a)] \tan\theta,$$
$$\text{lower west-east wall:} \quad y - b = (x - a) \tan\theta,$$
$$\text{outer north-south wall:} \quad y - (\cos\theta + \sin\theta + b) = -[x - (\cos\theta - \sin\theta + a)] / \tan\theta,$$
$$\text{inner north-south wall:} \quad y - b = -(x - a) / \tan\theta.$$
3. I devised an algorithm to locate the position (a, b) at which the largest number of nodes lie inside the corridors[1]. The algorithm is as follows. Start with an estimate of the best values of a and b. For $\theta = 0$ this is $(d - 1, 0)$ which places the starting rectangle as far down the west-east corridor as possible. For higher θ it is the optimum (a, b) from the previous θ. From this starting (a, b) take the nodes at $(a + ds, b)$ and $(a, b + ds)$ and find the numbers of nodes within the corridor boundaries when the inner corner is at each of these three positions. Regarding this number as a function of (x, y), fit a plane $Ax + By + C = 0$ to these three values. Its steepest gradient is in the direction $A\mathbf{e}_x + B\mathbf{e}_y$. Let P be the (a, b) position with the maximum node count so far. Advance from P in the direction $A\mathbf{e}_x + B\mathbf{e}_y$ by steps ds until the count of nodes within the corridors starts to fall. A new P may now have been reached. Repeat the process of taking two

[1] I had hoped to find a ready-made peak finding algorithm for data with two independent variable on the internet, but failed.

points adjacent to the new P, forming a second triangle, finding its plane and gradient, and stepping again up-hill. I allowed the process to continue until four triangles had been created, reducing the step length for triangles 3 and 4. Tests on a few continuous analytic functions such as $z = \cos x \exp(-2y^2)$ and some discontinuous but noise-free functions such as $z = INT[100 \cos x \exp(-2y^2)]$ showed that it worked sufficiently well.

4. The output to file has a header block listing parameters, a table of the optimum positions (a, b) of the inner corner in relation to the bottom centre of the object at $(0, 0)$ as a function of θ, a table of all nodes for which the flag $F = 0$, plus a smaller table listing the boundary nodes.

5. The boundary nodes noted in 4 are detected by scanning the grid to remove a) points for which the flag $F <> 0$, and b) nodes towards the centre of the object which are surrounded by nine other nodes each having $F = 0$.

4 Checks and error estimate

The tabulated results were processed to show the shape of the object and measure its area from the boundary points. For this I input the tables into Mathematica and plotted trends using Microsoft Excel. The square grid allows only an approximation of the area of the optimum object since all retained nodes lie inside the object or exactly on its boundary. Mathematica has a finite element package by which the table of (x, y) boundary values may be displayed and the enclosed area calculated. The `ToBoundaryMesh` function sorts the given table of nodes into a circuit of the perimeter and joins nodes around the perimeter with straight lines. The result still looks castellated as Figure 5 for a semicircle at grid spacing $0 \cdot 05$ shows.

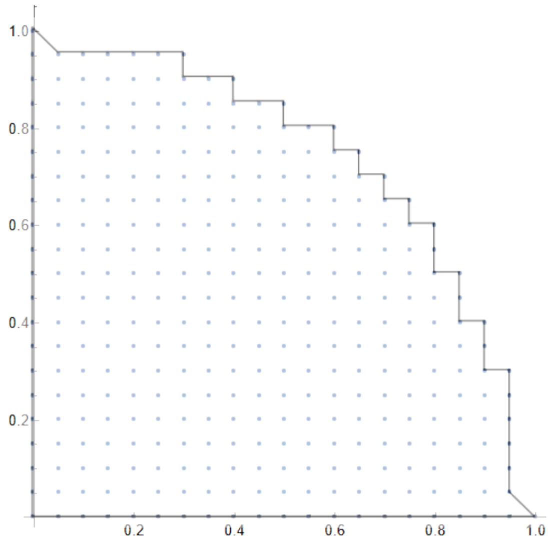

Figure 5: Right half of semicircle grid and its detected boundary calculated for $d = 1$, (a, b) fixed at $(0, 0)$ and node spacing $dx = dy = 0 \cdot 05$.

The semicircle also provides a guide to the underestimate of the object's area as a function of the grid node spacing $dx = dy$. As expected, the definition of the semicircle and its enclosed area both increase as the grid is made finer. Figure 6 plots the percentage error in the area of the semicircle as a function of dx. For example, Figure 5 is for $dx = 0 \cdot 05$ and the area enclosed by its jagged perimeter and that of its mirror half is $1{\cdot}475$ units, $6{\cdot}1\%$ less than $\pi/2$. This is plotted on the log-log Figure 6 as the second point down from top right. The grid spacing must therefore be less than $0 \cdot 005$ to obtain underestimates of no more than half a percent. I also found that decreasing the step in angle θ from $1°$ to $0 \cdot 5°$ made no difference to the shape or area, at least for the semicircle.

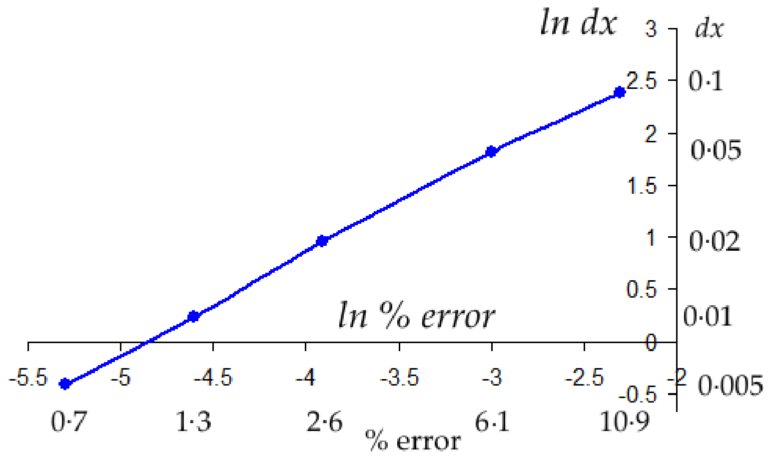

Figure 6: Log-log plot of percentage error as a function of node spacing $dx = dy$ in the grid.

5 Calculated shapes and areas

Figure 7 presents a sample of the shapes generated when d varies and (a, b) is required to migrate to the position for maximum area. Each object has mirror symmetry so only the right half is plotted. All have been calculated at the node spacing $0 \cdot 025$ and plotted using the `Region` function of Mathematica. The various values of starting half-length d produce a family of shapes. When d is almost $2{\cdot}3$, the sofa separates into disjoint parts, cut by the sharp inner corner. This shape is similar to that in the right panel of Figure 4, made by adding two feet to the semicircular arc, and which has $d = 1 + \sqrt{2} = 2 \cdot 41$. The shape at $d = 1$ also merits note because it is larger than and quite different from the semicircle in Figure 2. Figure 8 is a high resolution plot of this shape using the `ListPlot` function of Mathematica, with the semicircle superposed. The matching areas on the top outer edge more than compensate for area lost by the irregular notch around $(0, 0)$. This calculation gives its area as $1 \cdot 72$ to compare with $1 \cdot 57$ for the semicircle. Another notable and perhaps puzzling feature of these shapes is the cusps predicted along the lower surface; in few cases is this a smooth curve. This may in part be a consequence of limited resolution.

The half-length d of the initial rectangle clearly has a large influence on the size of the predicted object, as plotted in Figure 9. The blue points were calculated at grid spacing $0{\cdot}025$ and the green trend line fitted to values 3% larger, to allow for the likely under estimate of area, following Figure 6. The peak is near $d = 1 \cdot 4$ and it is tempting to suppose that it is precisely at $\sqrt{2}$, considering the example in Figure 4. On this assumption I have calculated the area for $d = 1 \cdot 415$ at grid spacing $0 \cdot 005$ and angle increasing by only $0 \cdot 5°$. The result

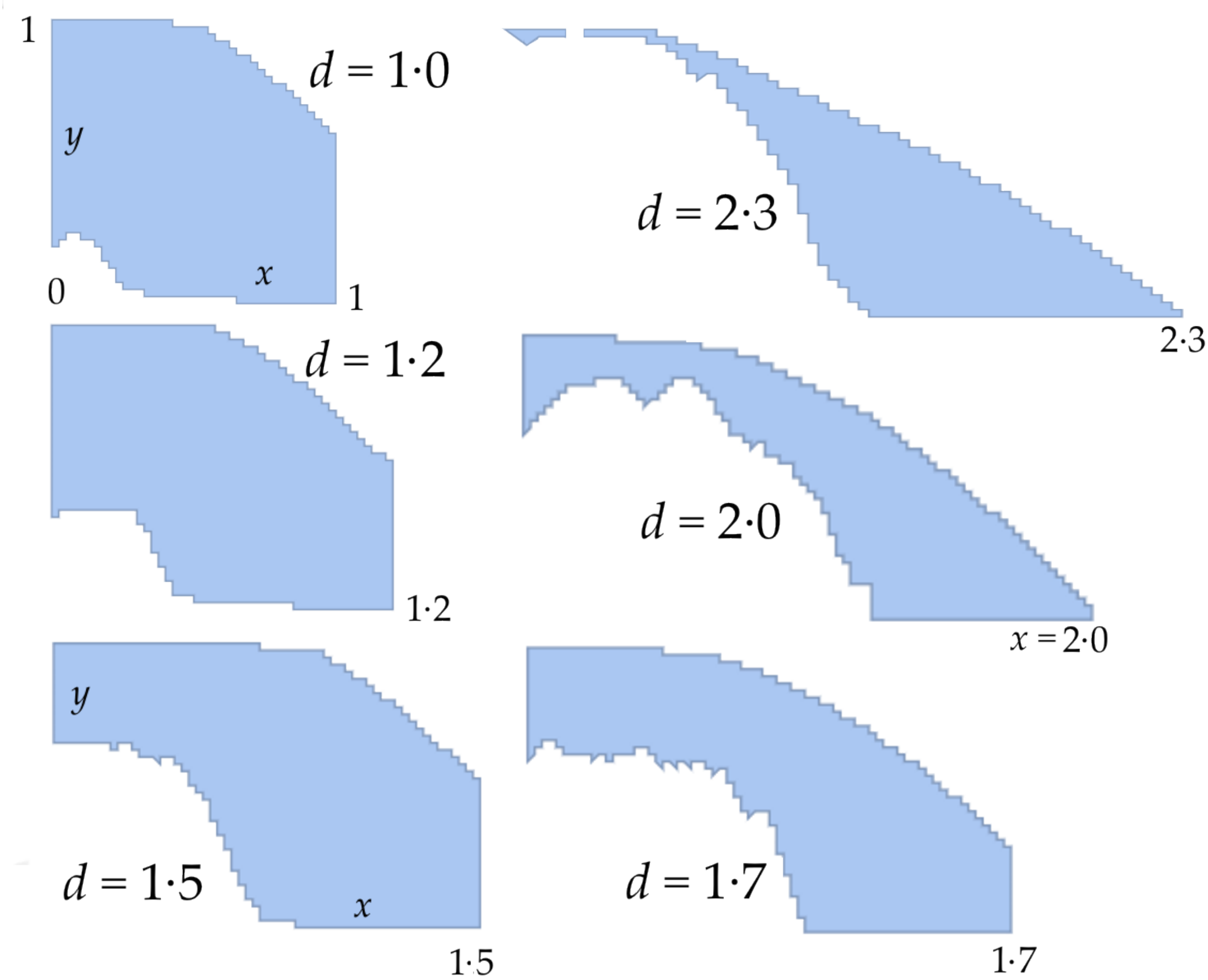

Figure 7: Examples of the predicted optimum shapes of object for various starting half-lengths d. Medium resolution with grid spacing $0 \cdot 025$, θ stepped in $1°$ intervals. Starting position of inner corner is $(d-1, 0)$.

is in Figure 10 where the left panel shows the grid and the right panel the region calculated by Mathematica from the boundary nodes. The area of the region is $2 \cdot 036$ square units but Figure 6 suggests that the true area is about 0.6% greater at $2 \cdot 048$. In summary, the algorithm has produced an object showing that $S \geq 2 \cdot 05$.

The left panel of Figure 11 shows how the inner concave profile of the sofa-object has been carved by the corridor's walls and in particular the sharp inner corner as it turns. The red curve is the locus of the corner at (a, b), and the blue curve is the final profile. Clearly the actual boundary of the sofa must lie either on or inside the cutting track of (a, b), and we see that, as θ has increased from 0 to 45°, the two curves converge. The arrows show increasing θ. The right panel shows the rate of carving as it turns by plotting the number of nodes deleted at each half-degree step from the starting number of 57084. The decrease is almost parabolic as the formula of the trend line shows. The decreasing number is consistent with the convergence of the red and blue curves in the left panel.

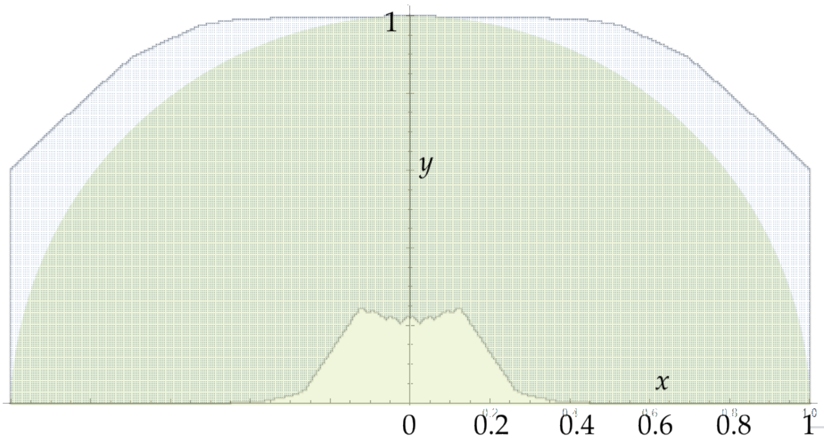

Figure 8: The left and right halves of sofa-objects with half-length $d = 1$. The semicircle is to be compared with the prediction of the algorithm plotted as its high resolution grid ($dx = 0 \cdot 005$).

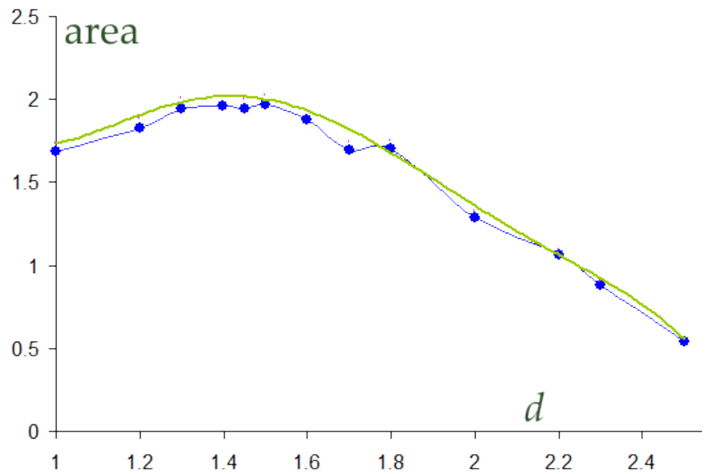

Figure 9: Area of sofa-object as function of initial half-length d. Calculated with $dx = 0 \cdot 025$. Green trend line augmented by 3%.

Further tests have been carried out to see whether the model is robust and consistently produces the same, of very similar, shapes and areas when parameters are changed. In one series of calculations the starting position of the inner corner (a, b) was varied from the $(d-1, 0)$ used for all calculations above. This was done only for the $d = 1 \cdot 415$ object with grid spacing 0·005 and half-degree steps in θ. Figure 12 shows the shapes predicted for $a = 0{\cdot}33$, $0{\cdot}41 = d-1$, $0 \cdot 5$, $0 \cdot 6$, $0 \cdot 7$ and $0 \cdot 8$, with $b = 0$. This is essentially a test of the area-maximising procedure. The results are remarkably stable for a from $< 0 \cdot 33$ to $> 0 \cdot 5$ and so give some confidence in the method. The cusp near $x = 0$ is present in all cases and is a puzzling outcome of the algorithm. The jagged inner boundary for large a will be a consequence of the maximisation procedure struggling in its hunt for positions of greatest area, and at some angles hacking off more than is necessary.

To summarise so far, starting from a rectangle the algorithm has produced a sofa-object with area $2 \cdot 05$ square units which certainly can be manoeuvred round the corner. The

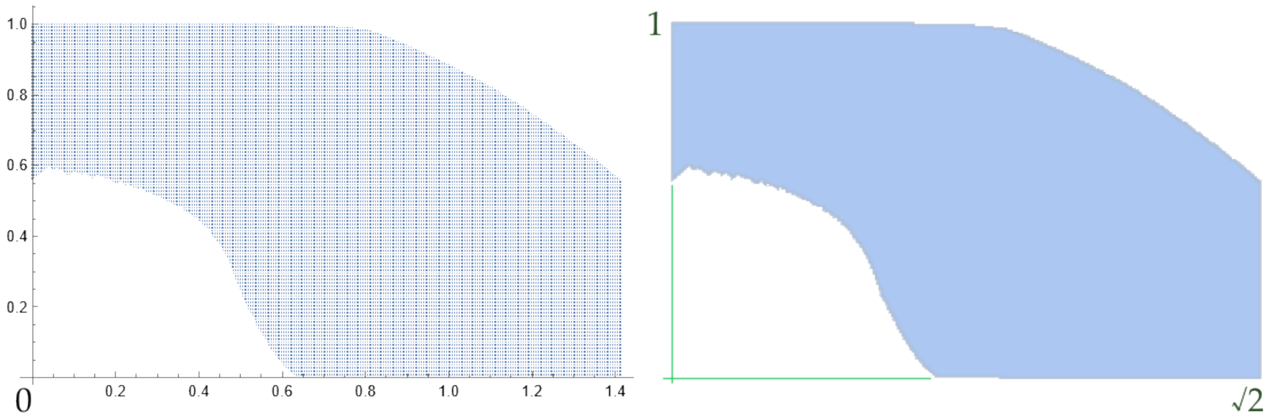

Figure 10: The largest object found by the algorithm, with $d = 1 \cdot 415 \approx \sqrt{2}$. Grid $0 \cdot 005$, θ increasing 0 to 45 by half degree steps. Area $2 \cdot 04$ square units. Left: the grid. Right: region found by Mathematica from boundary nodes.

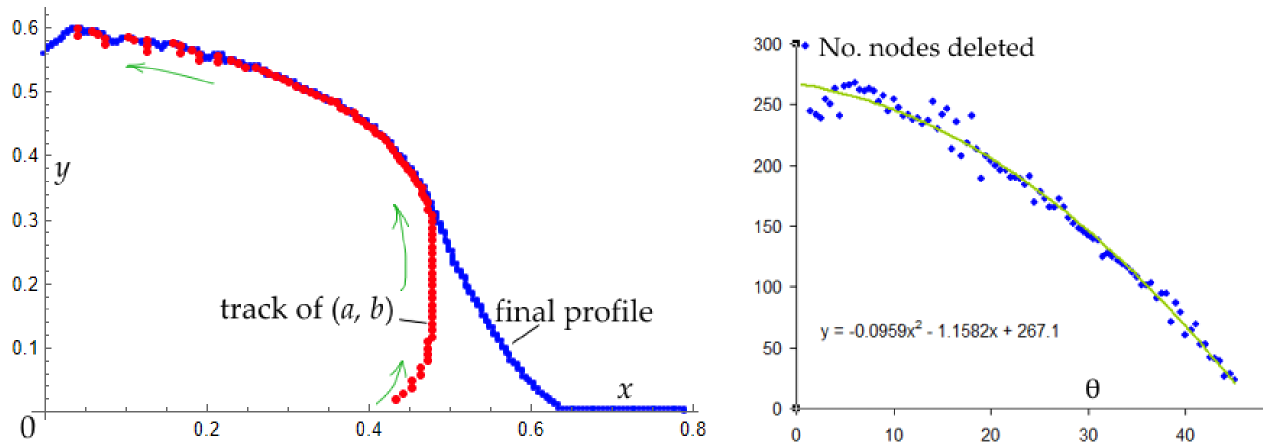

Figure 11: Development of the inner concave profile by removal of nodes. Left: track of inner corner and final profile. Right: number of nodes deleted at each half-degree of turn.

algorithm is quite intuitive since it simulates what one could do in practice with a wooden model using a saw, chisel or plane. The fact that it has produced a shape which is about 10% less than the suspected theoretical maximum for S suggests that the method has non-obvious constraints or blind spots. It is an example of a 'greedy' algorithm – one which grabs the best local or immediate choice at each stage, but cannot see the global picture. Consequently it will probably fail to find the overall best solution even though it is likely to find a fair solution in a fairly short time. My procedure for searching for the position of maximum area at any given angle is itself capable only of finding the nearest local peak, not necessarily the largest in the mathematical landscape.

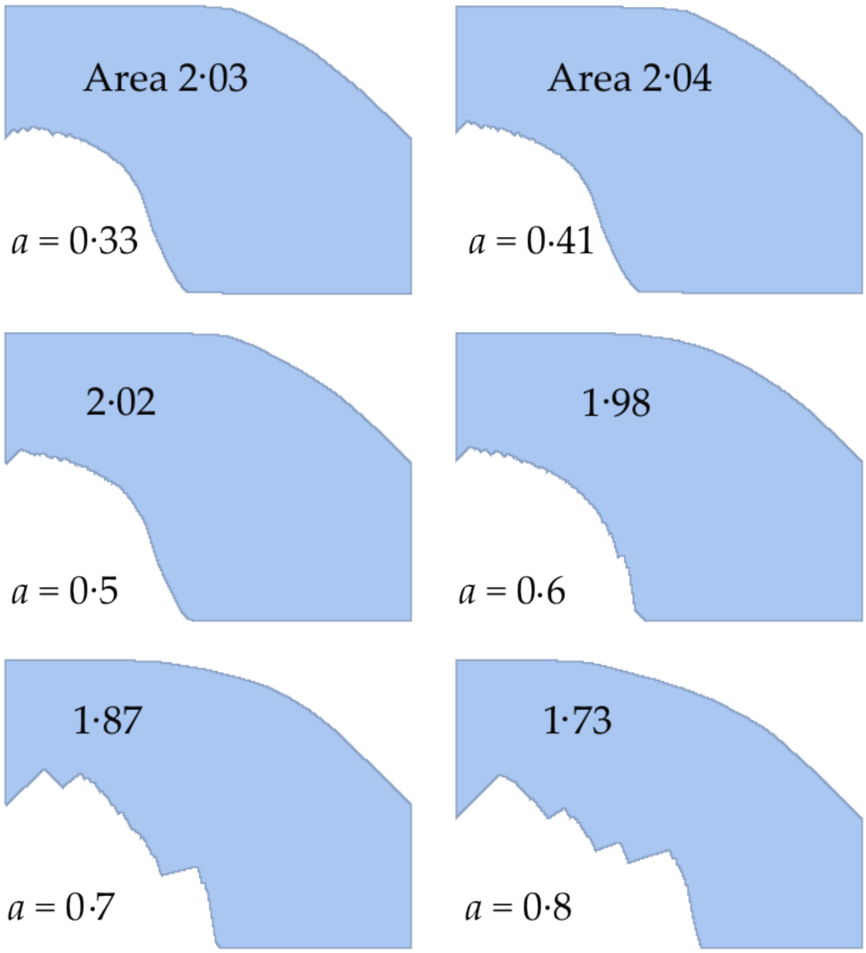

Figure 12: Effect on object shape and area of varying initial position $(a, 0)$ of the inner corner. All cases of $d = \sqrt{2}$, $dx = dy = 0 \cdot 005$, $d\theta = 0 \cdot 5°$.

6 By-hand adjustments

To confirm directly that the predicted object can be manoeuvred around the corner, I simulated its passage in a video animation made with Blender (*www.blender.org*). There is a link on *www.mathstudio.co.uk* from which six frames are reproduced in Figure 13. In making the video animation I adjusted the position of the sofa by hand and eye at selected angles and set key frames there. The animation not only shows that the computer-generated object can be turned, but makes clear that material can be added at the square ends A, some to the curved part of the upper edge at B, and some to build up the inner corners. I therefore turned from my program and from Mathematica to Blender and made small by-eye changes to the object, testing at each stage that it could still travel round the corner. Figure 14 shows the enlarged sofa. A second animation on *www.mathstudio.co.uk* shows this enlarged sofa, by now a tight fit. Its area can be measured in Blender using the MeasureIT add-on and is $2 \cdot 18$ square units. Figure 15 gives a selection of frames.

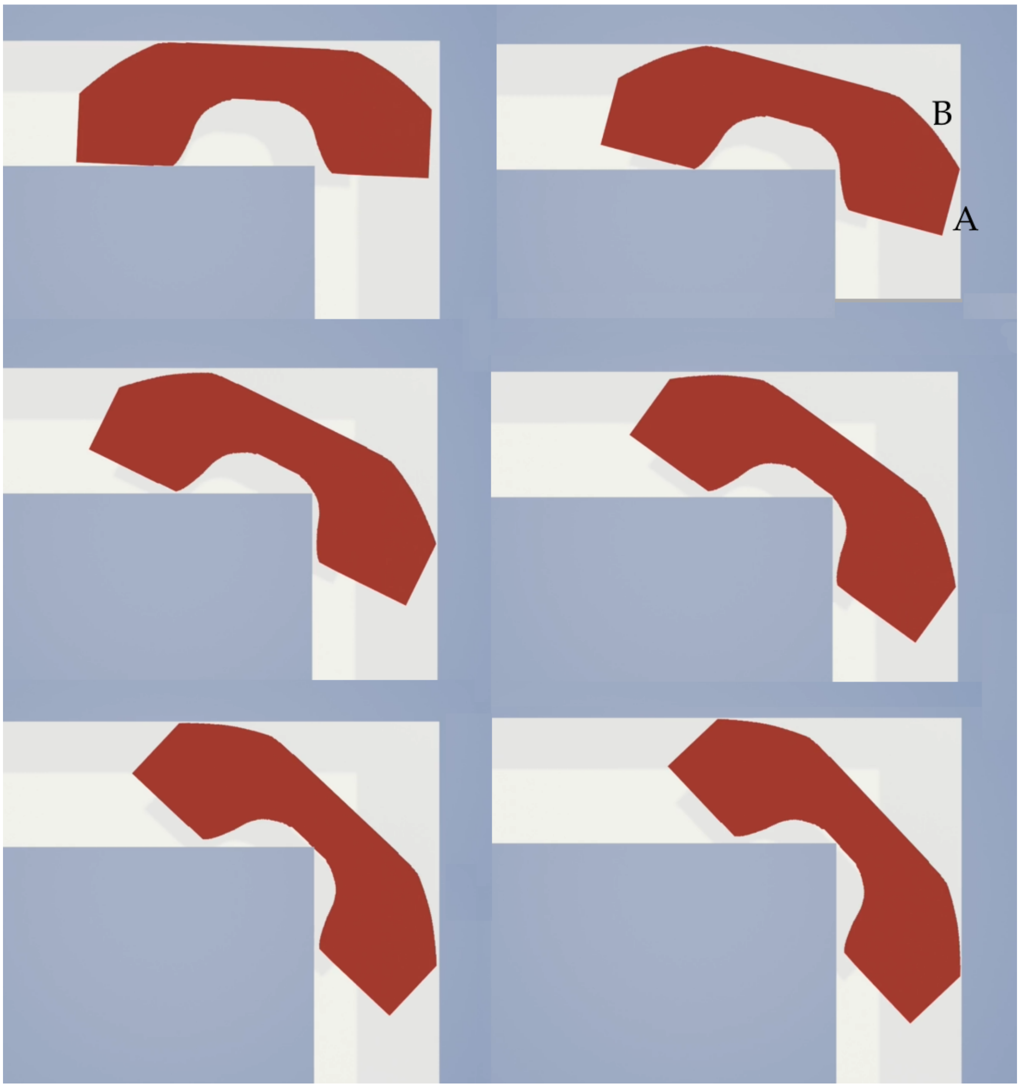

Figure 13: Six frames from the animation on *www.mathstudio.co.uk* showing the computer-generated sofa-object with area $2 \cdot 02$ turning in the corridor.

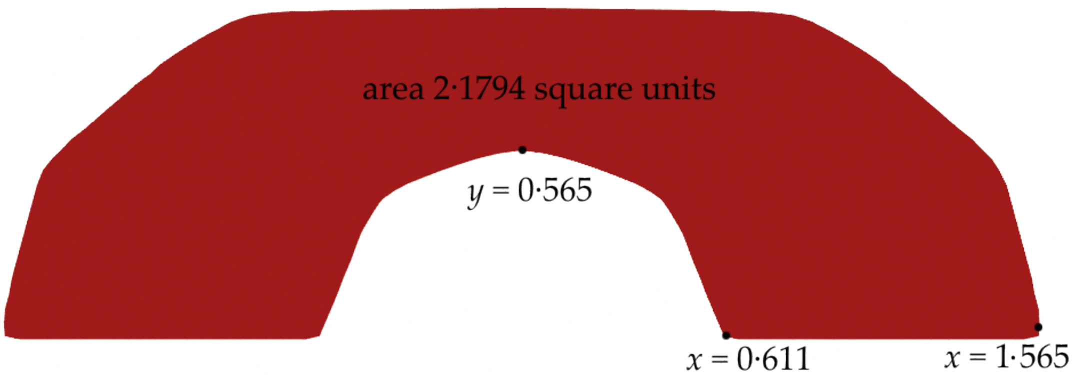

Figure 14: The sofa enlarged by hand to $2 \cdot 18$ square units.

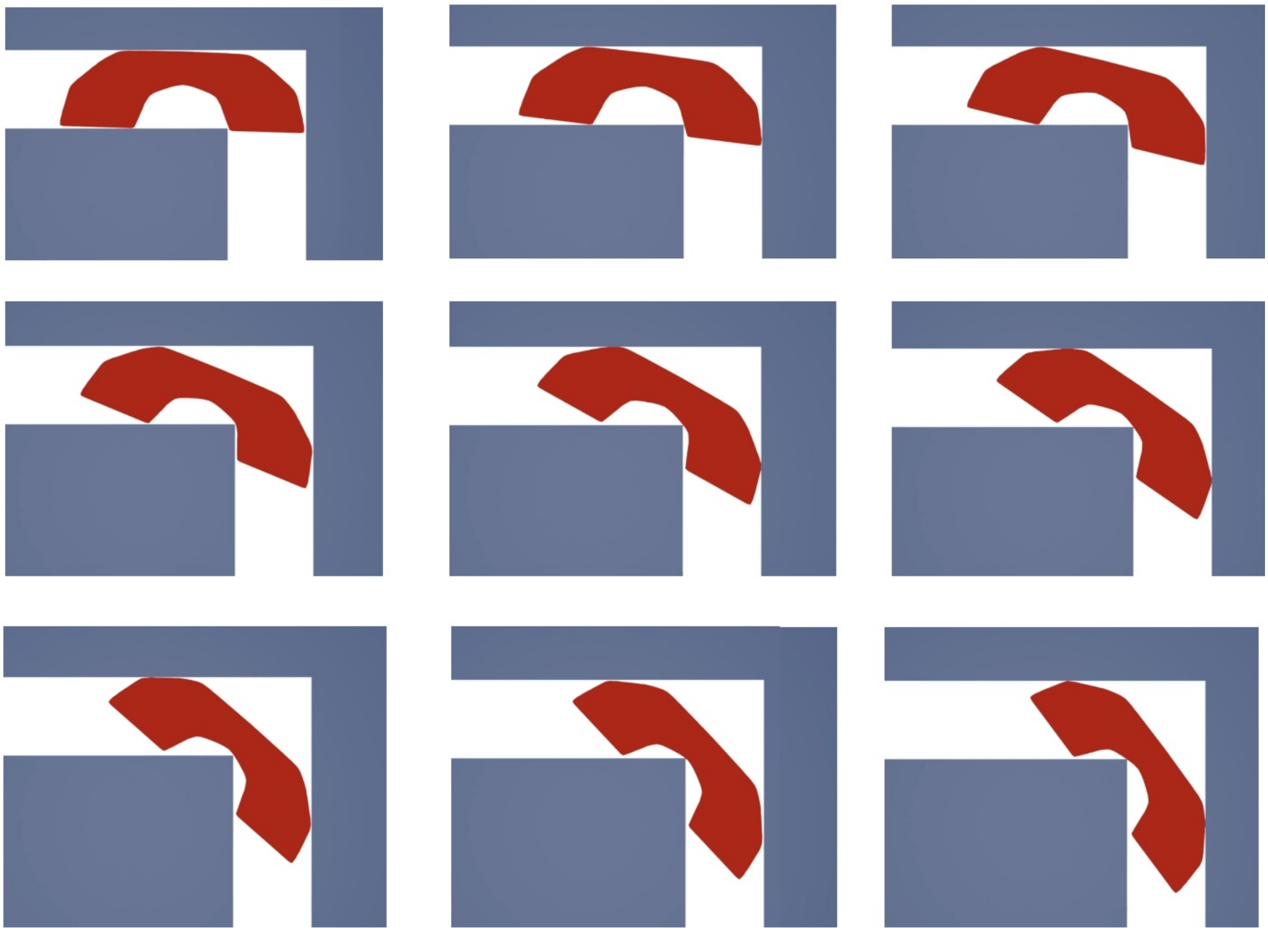

Figure 15: Nine frames from the second animation on *www.mathstudio.co.uk* showing the by-hand enlarged sofa with area $2 \cdot 18$.

7 The algorithm revisited

Having demonstrated the sofa in Figure 14 with its obliquely cut ends, I returned to the algorithm and started this time with a trapezium instead of a rectangle as the right hand half of the sofa. This is formed from the previous rectangle by cutting the right hand edge obliquely. Three parameters define the new figure and its starting conditions:

- the angle β at which the oblique end is cut. β is the angle away from the perpendicular to the base. In Figure 14 it is about 15 degrees,
- d, the length of the right hand half of the trapezium along its base from mirror line to the acute angled vertex on the right,
- a_0, the distance from the mirror line at which the right-angled corner in the corridor is positioned.

If $d > a_0 + 1$, the right hand point of the trapezium projects a little into the north-south wall of the corridor and therefore is immediately deleted. Thus the trapezium is blunted in order to gain area elsewhere.

Apart from these changes to initial conditions, the program implementing the algorithm runs in the same way and gives the same output, namely a list and count of all nodes inside or on the perimeter of the right-hand half of the sofa, plus a list of those just in the perimeter. I ran the program for a significant set of initial values of β, d and a_0 at a grid spacing of $0 \cdot 02$ in order to map out the 'landscape' of the count function. I then homed in on the combination giving the largest count and conducted a limited further search at grid spacing $0 \cdot 01$. The best combination was $\beta = 19 \cdot 0°$, $d = 1 \cdot 565$, $a_0 = 0 \cdot 49$. Figure 16 shows its perimeter. The internal node count was 10991 at spacing $0 \cdot 01$. 45 nodes were in the mirror plane, so the total number within the whole sofa was $2 \times 10991 - 45 = 21937$. In a separate article on *www.mathstudio.co.uk* I show that this node count is a reliable estimate of area for many planar figures, with an error of about ±5 cells, which is a trivial percentage error when there are 10,000 nodes. I therefore conclude that the area of the largest sofa found by the algorithm starting with a trapezium is $2 \cdot 194$ square units. This is an improvement on my by-hand version, coming closer to the limiting value – though the true value of this limit is still unknown. The algorithm by its nature guarantees that the sofa will fit into the corridor at all positions.

It seems natural to push the process further and use a slightly augmented version of Figure 16 (with all internal nodes too) as the starting object, so that the 'sofa' is developed by iteration and hopefully converges towards its limiting form. I have tried this idea by augmenting the best solution as found above with nodes one-deep around the perimeter at the curved sections of the shape. I ran the program again with this augmented version of Figure 16 as starting object and grid spacing ·02, and let the algorithm shave off nodes at each angle of rotation in the corridor as previously. The result was disappointing; although the previous best result could be recovered for a few particular values of d and a_0, I could not find conditions under which it was exceeded. This illustrates just how critically sensitive is the best shape. The exercise leaves the conclusion that Figure 16 with area $2 \cdot 194$ is the best found by this greedy algorithm.

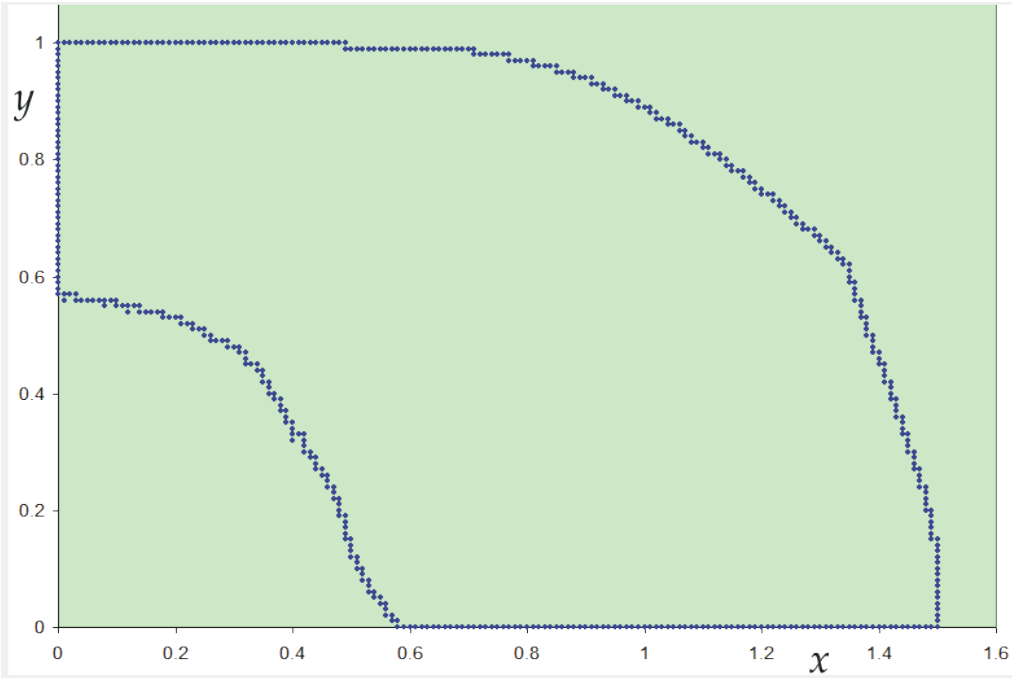

Figure 16: Right half of final 'sofa' produced by algorithm from an initial trapezium with $\beta = 19 \cdot 0°$, $d = 1 \cdot 565$, $a_0 = 0 \cdot 49$.

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