How far can a ladder reach up a wall?

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This started with a simple, almost trivial problem. I first heard of it when I was at secondary school and had just learned about Pythagoras' Theorem and quadratic equations. My friends and I tried but failed to solve it.

Q: Figure 1 shows a ladder of length h leaning against a wall, but constrained in how close it can come to the wall by a box (a coal bunker in my original version) which sticks out a from the wall and is b high at its front. How far can the ladder reach up the wall?

There is a subsidiary question. The man using the ladder, and working unobstructed by the bunker, has hung his paint pot by a hook on one of the rungs. If the ladder is initially vertical but slips down the wall right to the ground, what is the trajectory of the paint pot?

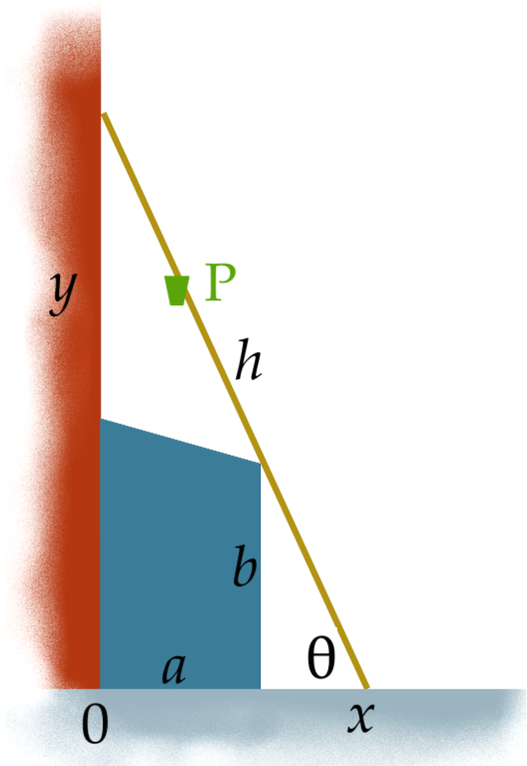

Figure 1: The ladder and paint pot P against the wall, obstructed by the coal bunker.

1 A quartic equation

We answer the main question first. Let the foot of the ladder be distance x from the wall and its top at height y. The ladder is inclined at angle θ from the ground. By Pythagoras' Theorem

$$x^2 + y^2 = h^2. \tag{1}$$

This would describe the situation if there were not the constraint of the box. With the ladder up against the box

$$\tan\theta = \frac{y}{x} = \frac{b}{x-a} \quad \text{so} \quad x = \frac{ay}{y-b}.$$

Therefore

$$\begin{aligned} \left(\frac{ay}{y-b}\right)^2 + y^2 &= h^2 \\ (h^2 - y^2)(y-b)^2 - a^2y^2 &= 0. \end{aligned} \tag{2}$$

This is a fourth order equation in y which cannot be reduced to a quadratic in y^2, so no wonder we schoolboys could not solve it. It is intriguing that a high order equation is needed to describe such a simple geometry. The ladder reaches y up the wall where

$$y^4 - 2by^3 - (h^2 - a^2 - b^2)y^2 + 2bh^2y - b^2h^2 = 0. \tag{3a}$$

Scaling the problem so that $h = 1$,

$$y^4 - 2by^3 - (1 - a^2 - b^2)y^2 + 2by - b^2. \tag{3b}$$

The equivalent equation in x is

$$x^4 - 2ax^3 - (1 - a^2 - b^2)x^2 + 2ax - a^2. \tag{3c}$$

As schoolboys we might by trial and error have found one or more numerical solutions for particular values of a and b. Being a quartic there are four roots, and in some cases 2 or 4 of them can be real. The table gives four real (x, y) pairs for $a = 0\cdot 2$, $b = 0\cdot 4$. x has been calculated from $ay/(y-b)$.

root	1	2	3	4
y	$-0\cdot 990$	$0\cdot 330$	$0\cdot 523$	$0\cdot 938$
x	$0\cdot 142$	$-0\cdot 944$	$0\cdot 853$	$0\cdot 349$

The four roots are illustrated in Figure 2. The wall is at $x = 0$ and ground at $y = 0$. The unit length of h for each solution is marked by a thicker line. Roots 3 and 4 have the front edge of the bunker/box (a, b) inside the line segment, while roots 1 and 2 place (a, b) on an extension of their respective lines. The lines for roots 2 and 4 appear to be at right angles to each other, but this is only approximately so – an insignificant coincidence. The two roots representing a real ladder are 3 and 4, with 4 reaching higher. Clearly these roots are interchanged if a and b swap sizes.

Suppose the ladder is up against the bunker and leaning on the wall. If a is fixed and b increased, the ladder will be pushed off the wall. Its foot then has to be adjusted to bring

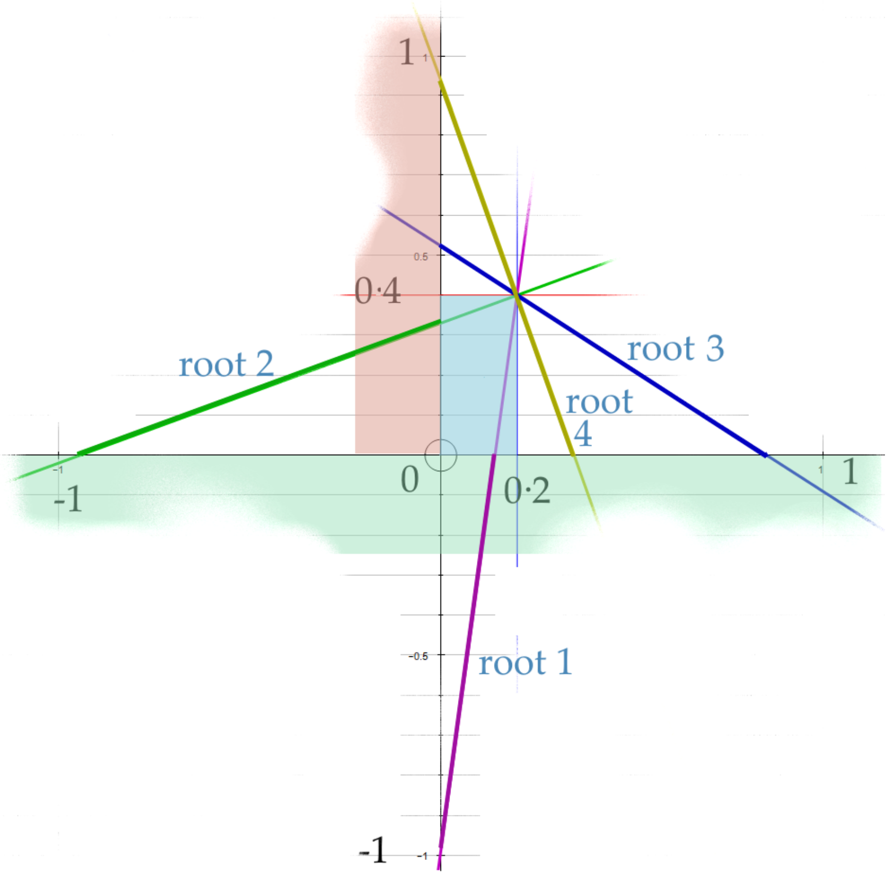

Figure 2: The four solutions to the quartic for $a = 0\cdot2$, $b = 0\cdot4$, $h = 1$.

it back onto the wall. This change in inclination is such that the higher y value of root 4 is decreased while the lower y value of root 3 is increased. If b is further increased, an inclination is reached at which the two roots coincide and the ladder is just long enough to maintain contact with both ground and wall. Any further increase in height of the bunker would push the ladder off the wall, and it would be too short to be brought back against the wall. If (u, v) is a point on the line segment of length $h = 1$ between wall and ground,

$$\frac{v}{u - x} = \frac{\sqrt{1 - x^2}}{-x}. \tag{4a}$$

(a, b) is on this line so

$$\frac{b}{x - a} = \frac{\sqrt{1 - x^2}}{x} \qquad \text{and} \qquad b_{max} = \frac{(x - a)}{x}\sqrt{1 - x^2} \tag{4b}$$

is the highest value b can take, for the given x and a, and the ladder just still touch both ground and wall. At the maximum value of b, $db/dx = 0$:

$$\frac{db}{dx} = \frac{a - x^3}{x^2\sqrt{1 - x^2}} \quad \text{so} \quad b_{max} \text{ occurs when } x = \sqrt[3]{a}. \tag{5}$$

For the ladder to reach the wall, the maximum dimensions of the bunker are related by

$$b_{max} = \frac{(a^{1/3} - a)}{a^{1/3}}\sqrt{1 - a^{2/3}} = (1 - a^{2/3})^{3/2}. \tag{6}$$

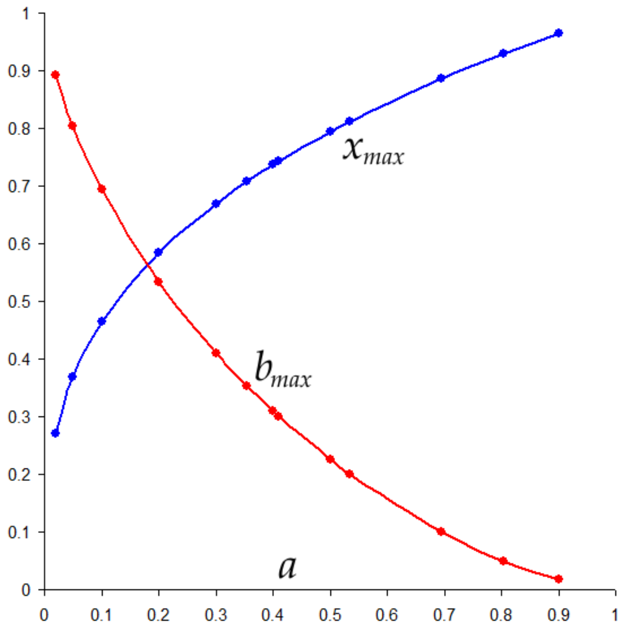

Figure 3: Values of box height b and position x of ladder foot for maximum reach up wall, as functions of box depth a. $h = 1$.

This function is plotted in Figure 3 together with the corresponding position x of the foot of the ladder. The $a - b$ curve is symmetric about the line $a = b$.

To summarise, if the bunker is larger than these limits, a ladder of length $h = 1$ cannot reach the wall. If smaller, it can reach the wall while leaning against the bunker, and can do this in either of two positions. The higher reach corresponds with root number 4 to Eq 3a.

2 Coincident roots

There can be four real roots only if the discriminant Δ of the polynomial satisfies $\Delta > 0$. The case $\Delta = 0$ occurs when at least two roots are coincident. Δ can be calculated from the Sylvester matrix. This is described in an article on Galois theory on *www.mathstudio.co.uk*, §13.2. It is the 7-by-7 matrix formed from the coefficients of the polynomial $P(y)$ and from those of its derivative dP/dy. As a monic polynomial

$$\begin{aligned} P(y) &= y^4 - 2by^3 + (a^2 + b^2 - h^2)y^2 + 2bh^2y - b^2h^2, \\ \text{and } P'(y) &= 4y^3 - 6by^2 + 2(a^2 + b^2 - h^2)y + 2bh^2. \end{aligned}$$

The Silvester matrix is

$$\begin{pmatrix} 1 & -2b & (a^2+b^2-h^2) & 2bh^2 & -b^2h^2 & 0 & 0 \\ 0 & 1 & -2b & (a^2+b^2-h^2) & 2bh^2 & -b^2h^2 & 0 \\ 0 & 0 & 1 & -2b & (a^2+b^2-h^2) & 2bh^2 & -b^2h^2 \\ 4 & -6b & 2(a^2+b^2-h^2) & 2bh^2 & 0 & 0 & 0 \\ 0 & 4 & -6b & 2(a^2+b^2-h^2) & 2bh^2 & 0 & 0 \\ 0 & 0 & 4 & -6b & 2(a^2+b^2-h^2) & 2bh^2 & 0 \\ 0 & 0 & 0 & 4 & -6b & 2(a^2+b^2-h^2) & 2bh^2 \end{pmatrix}$$

Taking $h = 1$, its determinant, which is the discriminant of $P(y)$, is

$$\Delta = 16a^2b^2(-a^6 - 3a^4b^2 + 3a^4 - 3a^2b^4 - 21a^2b^2 - 3a^2 - b^6 + 3b^4 - 3b^2 + 1) \quad (7)$$
$$= -16a^2b^2((a^2-1)^3 + (b^2-1)^3 + 3a^2b^2(a^2+b^2+7) + 1)$$
$$= -16(A+1)(B+1)[A^3+B^3+3(A+1)(B+1)(A+B+9)+1] \quad \text{where} \quad A = a^2-1, \;\; B = b^2-1,$$

a quartic symmetric in A and B. However $\Delta = 0$ is a cubic in A and B. The six roots for a, b of $\Delta = 0$ will each correspond to coincidence of one pair of the four roots of the quartic $P(y)$. We are most interested in the coincidence of roots 3 and 4, as in the table of §1. Note that since $a^2 < 1$ and $b^2 < 1$, $-1 < A < 0$ and $-1 < B < 0$. Δ will be zero when

$$G = A^3 + B^3 + 3(A+1)(B+1)(A+B+9) + 1 = 0. \quad (8)$$

The solution when $A = B$ is readily found:

$$2A^3 + 3(A+1)^2(2A+9) + 1 = (8A+7)(A+2)^2.$$

The root $A = -7/8$ corresponds to $a^2 = 1/8$, $a = 1/(2\sqrt{2}) = 0 \cdot 3536$ consistent with Figure 3. The other roots are complex. In the general case use Eq 6 to find that

$$B = (1 - a^{2/3})^3 - 1 = -3a^{2/3} + 3a^{4/3} - a^2.$$

Taken together with $A = a^2 - 1$ some involved manipulation can show that Eq 8 is satisfied. Since Eq 8 is symmetric in A and B, this also confirms the symmetry of the $a-b$ curve in Figure 3. Figure 4 was produced with Mathematica. It shows in orange the surface $G(A,B)$ and in blue the horizontal plane 0. The curved intersection is essentially the square of the $a-b$ curve in Figure 3.

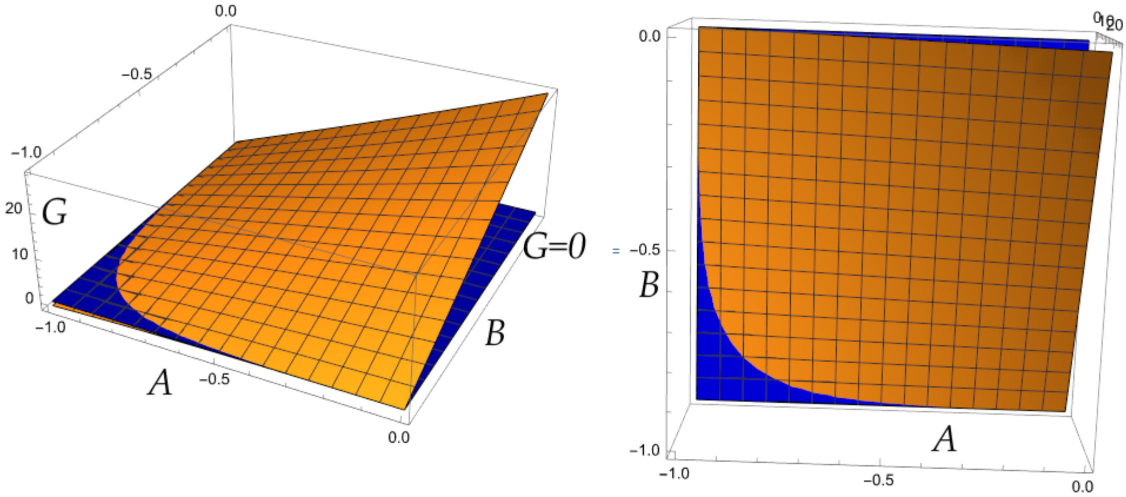

Figure 4: Two views of the cubic surface $G(A,B)$ and the plane at height 0, intersecting.

3 Some related geometry

In this section we return to the simple case where there is no coal bunker and examine the curve C which is the envelope of all positions of the ladder as it is moved up and down the wall as its foot is pushed along the ground. The ladder is thus regarded as being tangent to C at all angles of inclination.

3.1 Envelope curve: the astroid

Suppose the foot of the ladder is distance x from the wall, and let (u, v) be a general point along it. Let the equation describing its line be $L(x)$ where x is as a parameter setting the position of the ladder. There will be one point Q along the ladder at which the ladder is tangent to the envelope C. If x is moved a tiny distance δ, the line adjusts to $L(x+\delta)$. Q will be at the intersection of these two adjacent lines. Q is therefore found by solving $L(x) = L(x+\delta)$. The Taylor series begins $L(x+\delta) = L(x) + L'(x)\delta + ...$ so Q lies where $L'(x) = 0$. We already have the equation $L(x)$ at Eq 4a so

$$L'(x) = \frac{d}{dx}(u-x)\frac{\sqrt{1-x^2}}{-x} = \frac{u-x^3}{x^2\sqrt{1-x^2}}.$$

The reader will recognize this as precisely the formula defining b_{max}, Eq 5 in §1; the envelope is the $a-b$ curve in Figure 3. The reason is that the condition for two roots, numbers 3 and 4, to coincide, is that their respective ladder-lines become closely adjacent and eventually coincide, in which position they are tangent to the envelope and intersect on C. C is given in terms of parameter x by

$$u = x^3, \qquad v = (1-x^2)^{3/2}. \tag{9}$$

$$\text{or} \quad u = \cos^3 t, \qquad v = \sin^3 t.$$

By Pythagoras' theorem

$$u^{2/3} + v^{2/3} = 1.$$

The curve C is known as an astroid. The first person to write about it was Johann Bernoulli at the end of the 17th century. Some of its properties are described in the next two sections.

3.2 A rolling curve

The envelope of the sliding ladder has produced only one of the four sections of the astroid. It has four cusps which prompted its name as a star-shaped curve. It is an example of a roulette – a curve formed when one curve rolls along or around another. Specifically, it is the locus of a marked point on a wheel of radius r when it rolls without slipping on the inside of a circle whose radius is $4r$. The geometry is illustrated in Figure 5. The larger circle $\mathcal{B}$ is centred at the origin and has unit radius. The smaller circle $\mathcal{C}$ has radius $r = 1/4$. It has rolled on the inside of the larger circle through anticlockwise angle α and in so doing has rotated through angle $\beta = 4\alpha$ clockwise about its own centre O', which is at instantaneous position $(1-r, \alpha)$ in polar co-ordinates. When $\alpha = \beta = 2n\pi$, the point of contact between the circles is at $(1, 0)$, and when $\mathcal{C}$ has rolled one full turn, contact is at $(0, 1)$. We track the $u-v$ coordinates of that point P, fixed on $\mathcal{C}$, which lies on the u axis when $\alpha = 0$. The position of P with respect to axes through O' would be $(r\cos(\beta-\alpha), -r\sin(\beta-\alpha)$ in Cartesian co-ordinates. With respect to O, therefore, P is instantaneously at

$$u = r\cos(\beta-\alpha) + (1-r)\cos\alpha, \qquad v = -r\sin(\beta-\alpha) + (1-r)\sin\alpha, \qquad r = \tfrac{1}{4},\ \beta = 4\alpha. \tag{10}$$

Substituting for r and β gives

$$u = \frac{1}{4}\cos(3\alpha) + \frac{3}{4}\cos\alpha, \qquad v = -\frac{1}{4}\sin(3\alpha) + \frac{3}{4}\sin\alpha,$$

and using the expansions $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$, $\sin 3\alpha = -4\sin^3\alpha + 3\sin\alpha$ we recover $u = \cos^3\alpha$, $v = \sin^3\alpha$. This way of looking at the astroid makes clear how the cube powers arise naturally from the ratio of radii of circles $\mathcal{B}$ and $\mathcal{C}$.

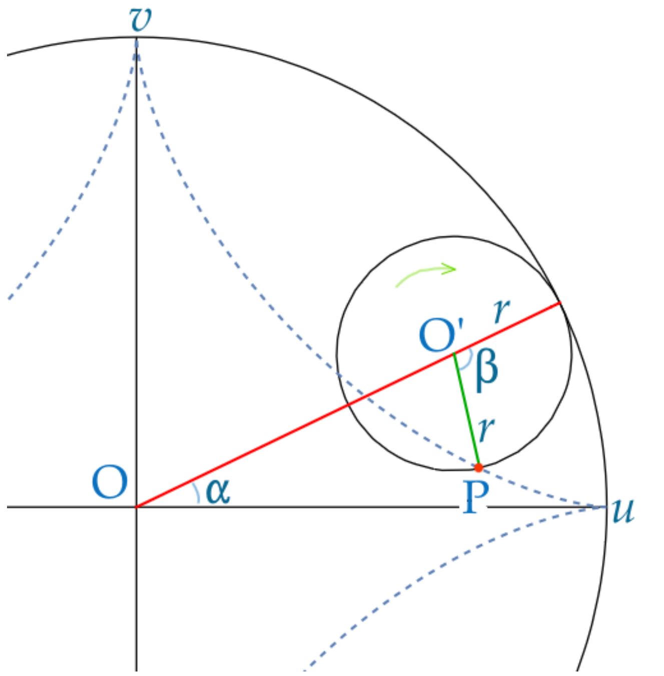

Figure 5: Generation of astroid by rolling of circle, radius $r = 0 \cdot 25$, inside circle radius 1.

3.3 A family of ellipses

The formula for the astroid has similarities with that of an ellipse, whose general formula is $u^2/c^2 + v^2/d^2 = 1$. In fact the astroid is not only the envelope of the sliding ladder, but the envelope of a family of ellipses which fit inside the four sections of the full astroid. To see this first find the unique ellipse whose centre is at the origin and which contains point $P = (p, q)$ where the gradient is m. Its gradient satisfies

$$\frac{2u}{c^2} + \frac{2v}{d^2}\frac{dv}{du} = 0 \quad \text{so at } P \quad \frac{1}{d^2} = \frac{-p}{mqc^2}.$$

$$\text{Therefore} \quad c^2 = p^2 - \frac{pq}{m}, \quad d^2 = q^2 - mpq.$$

Now set $(p, q) = (\cos^3 t, \sin^3 t)$ and $m = \tan t$. t is related to the inclination of the ladder in Figure 1 by $\theta = \pi - t$. (This ensures that m is negative in the first quadrant, a condition necessary for the curve to be an ellipse and not a hyperbola.) A selection from this family of ellipses, parametrised by angle t, is plotted in Figure 6.

3.4 An instrument for drawing ellipses

We turn at last to the second question about the trajectory of the paint pot as the ladder slips down to the ground. The situation is illustrated in the left panel of Figure 7. x is again

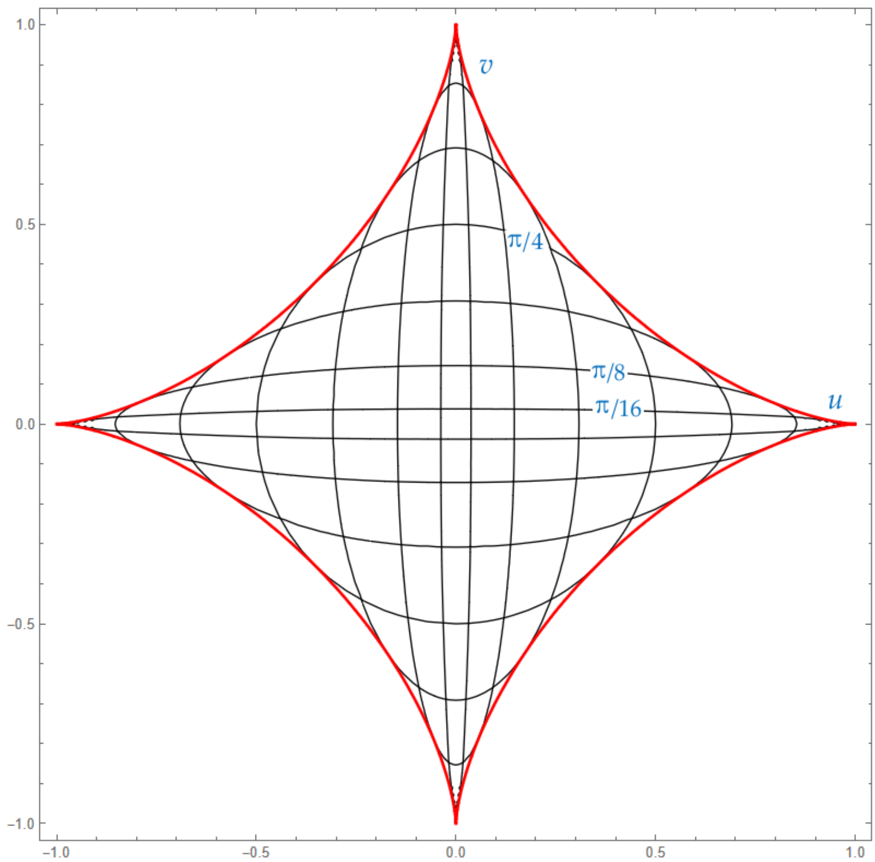

Figure 6: Family of ellipses contained within the astroid $(\cos^3 t, \sin^3 t)$ and tangent to it. Ellipses sampled in increments of t of $\pi/16$ from $\pi/16$ to $7\pi/16$.

a parameter and the ladder reaches $\sqrt{1-x^2}$ up the wall. The pot P is at a position which divides the ladder in the ratio $k : 1-k$, so it will be clear that when the ladder is vertical, the pot is at $(0, k)$, and at $(1-k, 0)$ when it lies on the ground. The general position of the pot is $(\,(1-k)x,\ k\sqrt{1-x^2})$. Plotting a few points for different values of x leads us to suspect that the locus of P is an ellipse with semi-axes $1-k$ and k. To prove this, consider the ellipse

$$\frac{u^2}{c^2} + \frac{v^2}{d^2} = 1 \quad \rightarrow \quad (1-k)^2 x^2 d^2 + k^2(1-x^2)c^2 - c^2 d^2 = 0.$$

This is satisfied if $c = 1-k$ and $d = k$. So the trajectory of paint pot on the slipping ladder is an ellipse. Remarkable.

The argument stands when P is at any point along the line of the ladder. Thus the configuration in the right panel of Figure 7 will have P describing an ellipse with horizontal semi-axis $1+k$ and vertical semi-axis k. Its position is like that of the point (a, b) for roots 1 and 2 in Figure 2. This is the basis of a simple device supposedly invented by Archimedes called his ‘trammel’ for drawing or cutting accurate ellipses. There are illustrations of it and similar ellipsographs on the internet. The wall and ground are replaced by two grooves at right angles in a wooden or metal block. The ladder is replaced by a straight bar to which two small rollers are attached at a fixed distance apart. One roller runs in the u groove and

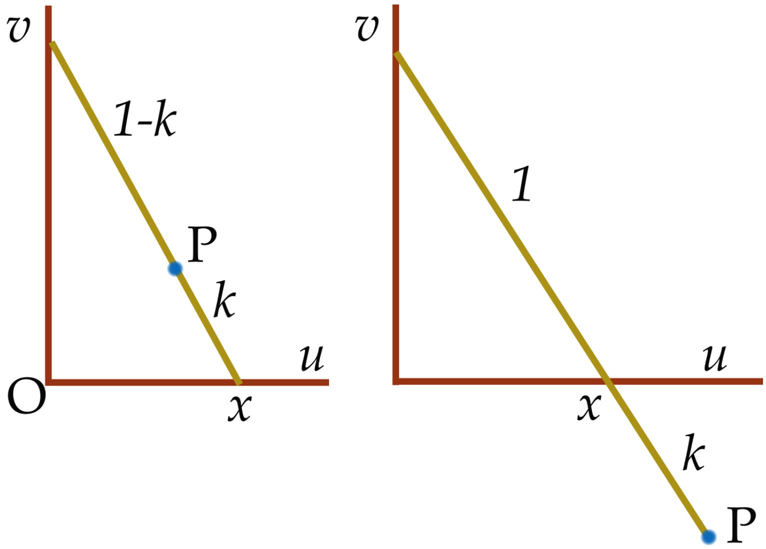

Figure 7: Left: Paint pot P clipped to the ladder in general position. Right: P on an extension of the ladder-line.

the other in the v groove, and the bar can swing around over the top of the grooved block subject to the constraint by the rollers. The pen or cutter is at a position along the bar, clear of the block, determined by the desired size and shape of the ellipse. Turning the bar causes the pen to draw the ellipse.

3.5 Some related curves

Many artistically attractive plane curves can be represented by formulae similar to those defining the astroid. Let us first convert Eq 9 into a polynomial in u and v. We have

$$u^{2/3} = 1 - v^{2/3} \qquad \text{so} \qquad u^2 = 1 - 3v^{2/3} + 3v^{4/3} - v^2,$$

$$u^2 + v^2 - 1 = -3v^{2/3}(1 - v^{2/3} = -3(uv)^{2/3},$$

$$(u^2 + v^2 - 1)^3 + 27u^2v^2 = 0. \tag{11}$$

This form for the astroid, a cubic in u^2 and v^2, has degree 6 and so 6 roots. If v is real and $0 < |v| \leq 1$, there is one positive real root for u^2 so u has two equal values of opposite sign. These define the astroid in the plane. If $1 < |v|$, there is one real negative root for u^2, so two purely imaginary values of u. The other four roots are of the form $u = \pm p \pm q$. The example illustrates that curves (called 'algebraic varieties') like the astroid have a structure beyond the Euclidean plane.

The first related curve is obtained from Eq 11 by changing the sign of $27u^2v^2$ from + to −. It is plotted in the left panel of Figure 8 together with the astroid (in green). The four petal-shaped lobes have cusps which meet the cusps of the astroid around the unit circle, so together the curves form a connected network of lines which could be the map of a race track or fairground ride.

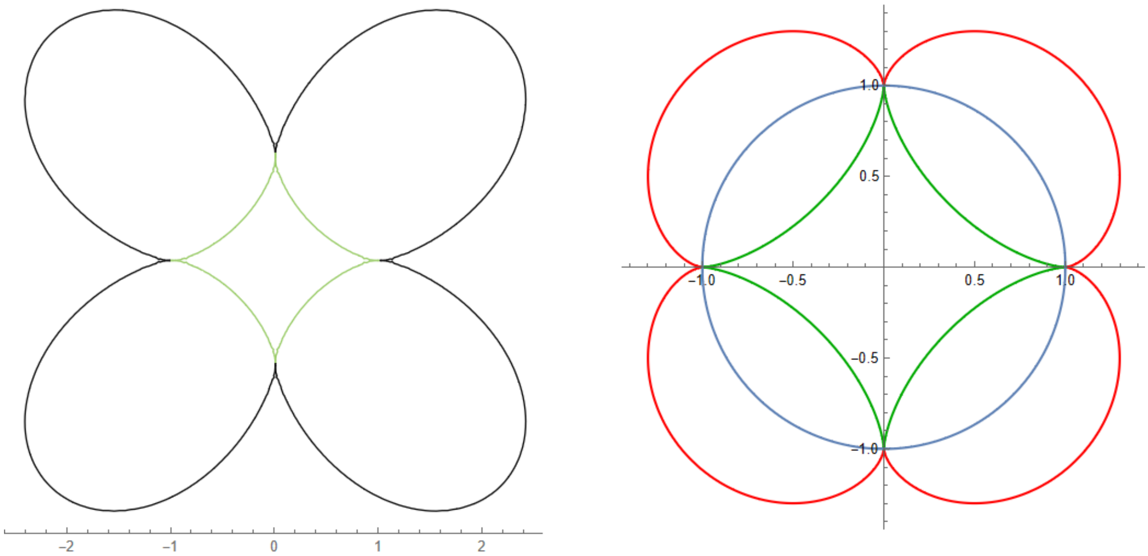

Figure 8: Two curves related to the astroid. Right: red curve is epicycloid.

The second curve is shown in red in the right panel of Figure 8. It is the epicycloid corresponding to the astroid being a hypocycloid – that is, it is the curve traced by a fixed point on a small circle, $r = 1/4$, which rolls on the *outside* of the unit circle (refer to Figure 5). The unit circle around which the small circles roll is shown in blue in the figure. Similar to Eq 10 for the astroid, the epicycloid is given parametrically by

$$u = -r\cos(\alpha + \beta) + (1 + r)\cos\alpha, \qquad v = -r\sin(\alpha - \beta) + (1 + r)\sin\alpha, \qquad r = \tfrac{1}{4},\ \beta = 4\alpha. \quad (12)$$

The two curves in Figure 9 have been obtained by changing some of the signs in Eq 10. On the left, looking like a slice of pre-sliced white bread, is

$$u = -r\cos(\beta - \alpha) + (1 + r)\cos\alpha, \qquad v = r\sin(\beta - \alpha) + (1 + r)\sin\alpha, \qquad r = \tfrac{1}{4},\ \beta = 4\alpha. \quad (13)$$

On the right is the slender and elegant

$$u = r\cos(\beta - \alpha) + (1 - r)\cos\alpha, \qquad v = -r\sin(\beta - \alpha) + (1 + r)\sin\alpha, \qquad r = \tfrac{1}{4},\ \beta = 4\alpha. \quad (14)$$

and more could be readily created and pleasingly decorated with colour.

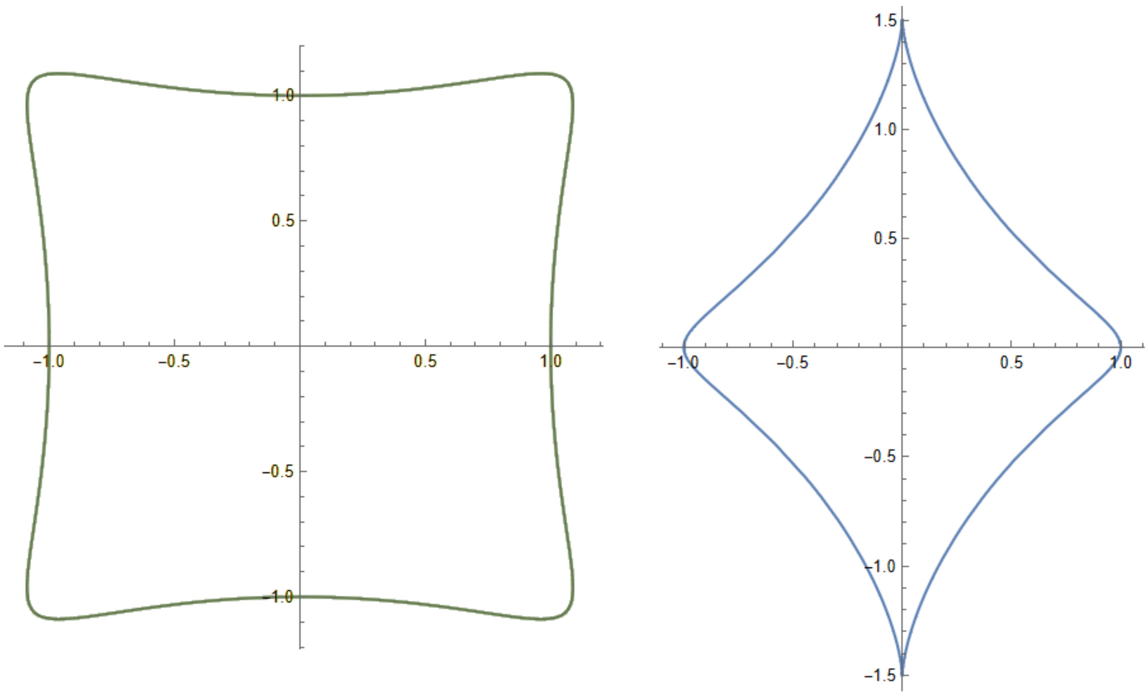

Figure 9: Two further curves with parametric equations similar to the astroid's.