Egyptian Fractions

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2023

1 Introduction

I came across this question in a maths paper for 12 year-olds:

> The ancient Egyptians used only 'unit fractions' which have 1 as their numerator. A general fraction was written as a sum of unit fractions such as $5/18 = 1/6 + 1/9$. Write $2/3$ as the sum of two or more *different* unit fractions.

Two answers are

$$\frac{2}{3} = \frac{1}{2} + \frac{1}{6} = \frac{1}{3} + \frac{1}{4} + \frac{1}{12}, \quad (1)$$

but are these the only ones? The question can be generalised :

1. Can every fraction n/d, with n and d coprime, be written as a finite sum of different unit fractions? If not, which can and why?

2. When a representation as a sum of unit fractions has been found, is it ever unique? If not, are there a finite or infinite number of representations?

3. Is there an efficient algorithm for finding all such unit fraction representations of a general fraction, and if so, what is it?

This article gives some thoughts on these questions.

2 Initial ideas

We can assume that $n < d$ since if $2d > n > d$, the unit fraction $1/1$ can be subtracted. Eq 1 can be immediately generalised by multiplying all denominators by an odd number k. Thus with $k = 3$

$$\frac{2}{9} = \frac{1}{6} + \frac{1}{18} = \frac{1}{9} + \frac{1}{12} + \frac{1}{36}.$$

We might hope, therefore, that any fraction with a composite denominator could have a factor divided out and so be reduced to a fraction whose unit representation has already been determined or is more readily found. If this were the case, fractions could be collected into classes, each class represented by the fraction with smallest and hence prime denominator.

It will be convenient to have a notation for these sums of unit fractions. I propose to write $1/a + 1/b + 1/c + ..$ as $[[a, b, c, ...]]$.

Turning the question round, what fractions are formed by adding 2, 3, 4 or more different unit fractions? Some examples are given in Table 1. The denominators of the contributing unit fractions are listed in each row and the resulting common fraction n/d is given in the last two columns. Thus $1/2 + 1/3 = 5/6$ and $1/3 + 1/5 = 8/15$. The table could be extended to millions of fractions, but begs the question "Can, say, 11/12 be so expressed?", 11/12 not being in the list[1]. One feature to note in Table 1 is how with many of the fractions n differs from d or $d/2$ by ±1. Of course, this may just be a spurious consequence of the few numbers chosen.

unit denominators						n	d
2	3					5	6
2	3	4				13	12
2	3	4	5			77	60
2	3		5			31	30
2	3	4		6		5	4
2		4				3	4
2		4	5			19	20
2			5			7	10
2				6		2	3
	3	4				7	12
	3		5			8	15
	3			6		1	2
	3				7	10	21

Table 1: Some sums of unit fractions.

An algorithm might develop along the following lines. Multiply n and d in turn by integer $k > 1$ to produce the equivalent fraction kn/kd. Take $1/(kd)$ as one term in the unit fraction decomposition, and look for a partition of $kn - 1 = a + b + c + ... + z$ such that each of a, b, ... z is a divisor of kd. For all the unit fractions to be different $1 < a < b < c < ... < z$. Taking 2/3 as an example and using $k = 2$, $k = 4$,

$$\frac{2}{3} = \frac{4}{6} = \frac{1+3}{6} = [[6,2]] \quad \text{and} \quad = \frac{8}{12} = \frac{1+3+4}{12} = [[12,4,3]].$$

The case $k = 3$, $2/3 = 6/9$, does not work since 6 does not split $1 + a + b...$ into a sum of different factors of 9.

Partitions of an integer in which all parts are distinct are naturally called 'distinct partitions' or 'strict partitions'. They were first systematically investigated by the 18th century genius Euler, and later by Ramanujan and Hardy. However, the constraints on the partitions for this problem are more severe in that we want only those that include 1. It serves our purposes to separate the distinct partitions of N into two classes: Class C1 is all those which have 1 as a part, and Class C2+ has all other parts, which necessarily start with 2, 3 or higher number. To illustrate this examine the partitions of 10 to 13 listed in Table 2 and separated into the two classes. It can be readily seen that C1 for N is the set C2+ for $N - 1$ with the

[1] It can : $11/12 = [[2, 4, 10, 20, 60]]$.

N	Class	partitions							
10	C1	1, 9	1, 2, 7	1, 3, 6	1, 4, 5				
	C2+	2, 8	3, 7	4, 6	2, 3, 5				
11	C1	1, 10	1, 2, 8	1, 3, 7	1, 4, 6	1, 2, 3, 5			
	C2+	2, 9	3, 8	4, 7	5, 6	2, 4, 5	2, 3, 6		
12	C1	1, 11	1, 2, 9	1, 3, 8	1, 4, 7	1, 5, 6	1, 2, 4, 5	1, 2, 3, 6	
	C2+	2, 10	3, 9	4, 8	5, 7	2, 3, 7	2, 4, 6	3, 4, 5	
13	C1	1, 12	1, 2, 10	1, 3, 9	1, 4, 8	1, 5, 7	1, 2, 3, 7	1, 2, 4, 6	1, 3, 4, 5
	C2+	2, 11	3, 10	4, 9	5, 8	6, 7	2, 3, 8	2, 4, 7	2, 5, 6

Table 2: Valid distinct partitions of integers 10 to 13.

unit 1 appended to each, and the partition $(1,\ N-1)$ included. In the table for each N the number of C1 and C2+ partitions is about the same. These small numbers compare with the large numbers of all partitions in which those with repeated parts are included.

Much has been published on partitions of integers, both the unrestricted and distinct types, and values are listed in Table 24.5 on page 836 of the Handbook of Mathematical Functions by Abramowitz and Stegun. There is a generating function for the number of distinct partitions:

$$\prod_{k=1}^{\infty}(1+x^k) \;=\; 1+x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+6x^8+8x^9 \tag{2}$$

$$+10x^{10}+12x^{11}+15x^{12}+18x^{13}+22x^{14}+27x^{15}+\ldots+448x^{33}+512x^{34}+\ldots+10,880x^{60}+\ldots$$

The coefficient of x^N gives the number of distinct partitions $q(N)$ of N. There is an asymptotic formula for this:

$$q(N) \;\sim\; \frac{3^{3/4}}{12\,N^{3/4}}\exp\left(\pi\sqrt{\frac{N}{3}}\right).$$

$q(N)$ counts the trivial partition N itself and of course includes both classes C1 and C2+. With unrestricted partitions there is a theorem that the number of partitions of N with no parts equal to 1 is $p(N)-p(N-1)$. Another theorem states that the number of partitions of N into distinct parts equals the number of partitions of N into an odd number of parts.

Looking at the row for $N = 12$ in Table 2, the partition (1, 5, 6) would allow the unit fraction decomposition of any n/d for which the augmented denominator kd is the lowest common multiple of 1, 5 and 6 (*i.e.* 30) or any multiple of this. Some cases are

$$\frac{2}{5} = \frac{12}{30} = [[5,6,30]], \quad \frac{2}{15} = \frac{12}{90} = [[15,18,90]], \quad \frac{2}{25} = \frac{12}{150} = [[25,30,150]].$$

Clearly, as discussed above, the latter two fractions are trivial variants of 2/5.

To summarise so far, where possible the given n/d could first be reduced to a standard representative by dividing out all but one prime factor of d to convert it to $n/p,\ p>n$. This may be helpful. Finding a unit representation of n/d will involve finding a multiplier k which transforms n/d into kn/kd such that there is a partition $(1, a, b, ...z)$ of kn in which each of a, b, .. z is a divisor of kd. This is fortunate since it severely limits the number of partitions of kn to be considered, and gives a fairly short list of possible numbers from which to build that partition. Interesting though the above information on integer partitions may be, hopefully we do not have to concern ourselves with it.

3 Towards an algorithm

Against the above background there is some prospect of developing an algorithm in the form of an informed trial-and-error search in which the partition of numerator kn is made solely of the factors of kd. As test cases, and to feel my way forwards, I will try to determine unit fractions representation of the continued fraction convergents of $1/\pi$, starting with the well-known 7/22 which we used at primary school.

Case 7/22 **and** 7/11 **:** In order for all parts of the partition of kn to divide the magnified denominator kd for some k, kd should be highly composite. For this reason I will try multiplying 22 by 15 so that $kd = 2.3.5.11 = 330$. This makes $kn = 105$, so all parts in the partition of 104 must divide 330. The available partitions parts made from products of these primes are therefore

$$2,\ 3,\ 5,\ 6,\ 10,\ 11,\ 15,\ 22,\ 30,\ 33,\ 55,\ 66,\ 110,\ 165.$$

It seems reasonable first to subtract the larger of these numbers from 104, then try to make up the remainder with the small numbers, much as one might choose to pay a bill with large then decreasing denomination bank notes before breaking out the small change. Proceeding on this basis

$$104 - 55 = 49, \quad 49 - 33 = 16 = 10 + 6.$$

We therefore arrive at

$$\frac{7}{22} = \frac{1 + 55 + 33 + 10 + 6}{330} = [[330, 6, 10, 33, 55]].$$

Let us take a similar approach to 7/11. Multiplying by $k = 2.3.5 = 30$ gives $kn/kd = 210/330$ together with same set of factors. Then

$$209 - 165 = 44 = 22 + 11 + 6 + 5 \ \text{ so } \ \frac{7}{11} = \frac{1 + 165 + 22 + 11 + 6 + 5}{330} = [[330, 2, 15, 30, 55, 66]]$$

and so $7/22 = [[660, 4, 30, 60, 110, 132]]$.

Was this beginner's luck? A more severe test is 113/355, the fourth continued fraction convergents of $1/\pi$. 113 is prime and $355 = 5.71$. This can be reduced to $(1 + 42/71)/5$, so can we even represent 42/71?

Case 42/71 **and** 113/355**:** Deal with 42/71 first. In order to make its denominator highly composite, try again multiplying by $k = 30$, giving $kn = 1260$. The available factors of $kd = 2130$ are:

$$2,\ 3,\ 5,\ 6,\ 10,\ 15,\ 30,\ 71,\ 142,\ 213,\ 355,\ 426,\ 710,\ 1065. \tag{3}$$

The reduction sequence $1259 - 1065 = 194$, $194 - 142 = 52$, $52 - 30 = 22 = 15 + 5 + 2 = 0$ works and gives

$$\frac{42}{71} = \frac{1 + 1065 + 142 + 30 + 15 + 5 + 2}{2130} = [[2130, 2, 15, 71, 142, 426, 1065]].$$

Using $113/355 = (1+42/71)/5$ we also obtain $113/355 = [[5, 10, 75, 355, 710, 2130, 5325, 10650]]$. The gaps between the larger factors at Eq 3 are so large, and the gaps between the smaller one so small that I think this is the only partition of 1260 containing 1 which can be formed from these numbers. This situations will arise whatever primes form the denominator kd.

We seem to have the makings of an algorithm. To test it further here is a decomposition of 113/355 using $k = 2.11.23$, giving $kn/kd = 57178/179630$. Available factors for the partition are

$$2,\ 5,\ 10,\ 11,\ 22,\ 23,\ 46\,55,\ 71,\ 110,\ 115\,142\,230,\ 253,\ 355,\ 506,\ 710,\ 781,$$

$$1265,\ 1562,\ 1633,\ 2530,\ 3266,\ 3905,\ 7810,\ 8165,\ 11730,\ 17963,\ 35926,\ 89815.$$

Subtract in turn the largest number less than the numerator. Thus

$$57177 - 35926 = 21251, \quad 21251 - 17963 = 3288, \quad 3288 - 3266 = 22$$

which itself is a factor. Hence $113/355 = [[5, 10, 55, 8165, 179630]]$.

It is clear that unit faction decompositions are not unique for all n/d. Indeed, since there are infinite choices of the multiplier k, there are probably a large number of decompositions for each n/d, in which case the shortest one would probably be preferred. I see no way to identify the shortest decomposition other than by searching, but am content to leave that question to the fabled 'interested reader'.

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