

'PermGroups + WordGroups'
A Teaching Calculator for Finite Mathematical Groups
A pair of elementary interactive computer programs to support
self-study in algebraic group theory, including character tables.

Notes on use and examples for self study

John Coffey, Cheshire, England, 2009
Updated 2017

Preface

This note is about learning elementary group theory – the mathematical analysis of symmetry. It describes a pair of complementary computer programs called 'PermGroups' and 'WordGroups'. Each is a basic group calculator suitable for undergraduate study. By experimenting with the programs and following the examples described in full in these notes, I hope you will become hands-on familiar with some finite groups and their main features. This may help you with university maths exams, and may even bring some interest and pleasure!

'PermGroups' and 'WordGroups' are essentially two versions of the same program; they differ only in the method used to create and represent the group. 'PermGroups' uses permutations and 'WordGroups' uses abstract word presentations. Both generate finite groups of small order, calculate products of elements, conjugacy classes, commutators, subgroups and their cosets. There is the option to restrict study to this introductory level material or to include the more advanced topics of character tables and representation matrices, including characters lifted and induced from a subgroup. The programs together can therefore be matched either to a Year 1/Year2 course, or to a Year3/Year 4 course.

Both programs are interactive to a limited extent through Yes or No answers to questions printed to screen, though the presentation of output is crude. I give no guarantees of performance, but other undergraduate students may find them helpful to support self-study. They are offered in this spirit.

My motivation for writing these programs was as follows. I studied maths at university, and although I found group theory interesting, I struggled at times with some of the abstract concepts, largely because I did not have a 'feel' for what groups were like and how they behaved. Contrast this with geometry. We get a feel for two and three dimensional geometry at an early age by drawing rule and compass constructions and solving simple geometrical theorems. But groups?! Apart from simple rotational and reflection properties of regular polygons, it is not obvious how to gain hands-on experience with groups. I found almost all the textbooks short on illustrations and mostly heavy going. This was my motivation for writing this computer program. I offer it to other students who learn kinaesthetically and need some groups to experiment with. Those of you who are wholly comfortable with abstract thought in algebra need read no further.

This note, therefore, describes two companion computer programs and, by examples of increasing complexity, uses them to introduce elementary group algebra and character theory. It therefore complements introductory textbooks.

In Part 1 is a statement of what the two programs do.

Part 2 is about using 'Word' Groups and contains ten examples to get you used to the program and explore groups using it.

Part 3 is about using 'PermGroups'. Since PermGroups is essentially the same as WordGroups in its calculations and analysis, discussion in this Part is limited to how PermGroups differs, so please also read the more detailed account in Part 2. However PermGroups is more suited to generating the symmetric and alternating groups, S_n and A_n , so there are more examples of these in Part 3 than in Part 2.

There are some very sophisticated group theory programs available over the Internet, such as GAP and Magma. These are research tools for the experienced. I hope my simple programs will help you graduate to using these.

John Coffey, February 2009

Index

	Page
PART 1: Outline of what the two programs do	4

PART 2 : Worked Examples using WordGroups

This Part contains quite a lot of explanation of how to use the program, and the examples increase in complexity to provide a structured programme for self study.

2.1 Not a finite group : See also Appendix	11
2.2 Cyclic group, C_5	12
2.3 C_{12} and abelian direct products, $C_4 \times C_3$	14
2.3.1 C_{12}	14
2.3.2 $C_4 \times C_3$	16
2.4 Direct products $C_2 \times C_2$, $C_3 \times C_3$ and $C_4 \times C_4$	17
2.5 A symmetric group $S_3 =$ a dihedral group D_3	19
2.6 Another dihedral group, D_4	24
2.7 Comparison of D_4 (dihedral) with Q (quaternion)	29
2.8 An abelian semi-direct product $C_3 \wedge C_4$	30
2.9 An alternating group, A_4	33
2.10 The symmetric group, S_4	35
2.11 The five groups of order 20.	40

PART 3 : Worked Examples using PermGroups

Note that the description in Part 3 is scant, dealing mainly with the use of PermGroups. Most of the technical explanation is in the corresponding sections in Part 2.

3.1 A cyclic group, C_5 (Z_5)	42
3.2 Abelian direct products, $C_m \times C_n$	44
3.3 A symmetric group $S_3 =$ a dihedral group D_3	44
3.4 S_4 and its subgroups A_4 and D_4	45
3.5 A non-abelian direct product $S_3 \times C_3$.	48
3.6 Symmetric group S_5 and alternating group A_5	51

Appendix :

Table of orders and relation $\langle ba = \text{'word'} \rangle$
which generate finite groups using WordGroups

Index of Group Theory topics

These notes are definitely not a text book on group theory, but the following topics are illustrated in the text and by use of the programs:

	Page
abelianisation factor group	6, 19
action of a group	5
automorphism, $\text{Aut}(G)$	26
centraliser	5, 12
centre, Z	5, 12
character	7
commute	5
commutator	5, 13
complete group	28
composition series	37
conjugate	5
conjugacy class	5
coset	13
cyclic group	12
cycle notation	9
derived (commutator) subgroup	5, 13
direct product	15, 44, 48
factor group	6
faithful representation & character	7, 20
induced character	8, 20
inner automorphism, $\text{Inn}(G)$	26
irreducible representation & character	7, 23
isomorphism	18, 19
Lagrange's theorem	14
lattice diagram of subgroups	8, 26
lifting, lifted character	20
Masche's theorem	23
matrix representation	8
normal series	37
normal subgroup	15, 37
orbit-stabiliser theorem	25, 34
orthogonality of characters	20, 28
partition of integer N in S_N	36
permutation matrix and character	40, 45, 46
permutation notation	9
presentation of a group	4
regular representation & character	29
restriction of character to a subgroup	47
semi-direct product	20, 30
similarity transformation	23
simple group	15, 51
solvable group	37
Sylow theorems	42
symmetry of tetrahedron	33
transposition	9
transversal	15
trivial representation and character	7, 14

PART 1: Outline of what the two programs do

Introductory Group Theory Material

Steps 1 to 6 below cover basic group structure and exclude representation and character theory, so could be used to support early study in the basics of group theory. Either program is restricted to this option when the user selects 'n' to the Advanced Option prompt.

Step1 of the program PermGroups is different from WordGroups, but otherwise the two do almost exactly the same.

Step 1a) : Create a finite group with WordGroups.

WordGroups generates finite groups from 'words' made from two generators, **a** and **b**. You can think of **a** and **b** as representing physical operations such as rotation or reflection. Copies of the two letters **a** and **b** are joined together into arbitrarily long 'words' by concatenating all possible finite strings of **a** and **b**, such as **aabbbba** or **baabbbbaaaaab**. You also specify at least three constraints on **a** and **b**; this is called a 'presentation' of the group. Two of these constraints are the orders of **a** and **b**: that is the power of **a** or **b** which equals the group identity, I. For instance $\mathbf{a}^4 = \mathbf{a}^4 = \mathbf{I}$, which could correspond to **a** representing a rotation through a right angle about a fixed axis. You must also give at least one 'relation' linking **b** to **a**. Usually this will be $\mathbf{ba} = \mathbf{a}\dots\mathbf{b}$ where the dots stand for some combination of **a** and **b**. Internally the program stores the group elements as strings, with the empty string being the group identity. Thus $\mathbf{a}^4 = \mathbf{a}^4 = \mathbf{I}$ would be stored as 'aaaa' = ''.

An example of a group presentation as written in the literature is $\langle a, b \mid a^4, b^2, ba = a^3b \rangle$, meaning there are two generators, a and b, with $a^4 = b^2 = \text{Identity}$, plus one relation $ba = a^3b$.

Once you have entered the order of the elements and the relations, WordGroups

- i) forms increasingly compound concatenations of **a** and **b**,
- ii) reduces the 'words' to a standard form by replacing \mathbf{a}^n with I, \mathbf{b}^m with I and \mathbf{ba} with $\mathbf{a}\dots\mathbf{b}$ respectively.
- iii) removes duplicates, looks at inverses and products of the reduced words and generally checks for consistency. It turns out that for some selections of $\mathbf{a}^n = \mathbf{I}$, $\mathbf{b}^m = \mathbf{I}$ and $\mathbf{ba} = \mathbf{a}\dots\mathbf{b}$ only a finite number of independent 'words' can be formed and these behave as a group under the group operation of concatenation.
- iv) lists the elements and their inverses.

So we have created a group. The number of elements $|G|$ is the order of the group, G. The program calculates multiples of each element and hence finds its order. The number of elements with each order is tabulated.

A word of warning here: the 'word problem' is a knotty issue in advanced group theory. It concerns whether it is possible to be sure that two 'words' really represent different group elements. Cases are known where this cannot be decided even in principle, though we are not likely to meet any here. It is possible to input relations into WordGroups which will push it beyond its limits of validity – for instance, trying to generate an infinite group. The examples given in Part 2 of this note are all genuine finite groups.

WordGroups is restricted to :

- only two generators,
- only three additional relations,
- a recursion depth of at most four in forming words.

For some groups the 'word' presentation used here is more straightforward than the permutations of PermGroups, and for others the permutation representation is more powerful or illuminating.

Step 1b) : Create a finite group with PermGroups

PermGroups generates finite groups from permutations of N user-specified symbols. The groups it creates are, therefore, either the symmetric group S_N of all permutations of N elements, or one of its subgroups. You seed one or more permutations and the program calculates all their powers, products and their inverses until no more new elements are generated. This should ensure that the set of elements is closed under composition of elements. The program checks that the identity permutation is created, as required for a group. Concatenation of permutations is associative, so meeting another criterion for a group. A finite group is thereby generated.

Step 2): Find elements which commute.

Two elements, f and g , of group G commute if the ‘products’ fg and gf are equal, where product here is string concatenation. Commuting is one of the essential concepts in group theory. Physically it means that the order in which two operations are performed does not matter; the outcome is the same.

Either program takes each element g of G in turn and finds all the elements which commute with it. For each g the set of such commuting elements is called the ‘centraliser’ of element g . This set, in fact, forms a group under concatenation; hence this is a procedure for finding some of the subgroups H of G. The identity element, I, of course, commutes with every element, but in some groups there are other elements which share this property. The set of these all-commuting elements is called the ‘group centre’, Z. The program lists the elements in Z.

Step 3): Find the conjugacy classes.

Conjugacy of elements is another key concept in group theory. It is a weaker form of commuting. We say that f and g are conjugates if there is at least one element t for which $ft = tg$. This is usually written in the form $t^{-1}ft = g$, or equivalently $tgt^{-1} = f$, where the pair t and t^{-1} are an element and its inverse.

Another way of looking at conjugation is to see it as one manner in which a group can act upon the set of its own elements. The concept of a ‘group action’ is fundamental in group theory. We can think of a function $T(f)$ meaning $f \rightarrow t^{-1}ft$ where t is another group element and the operation is multiplication (concatenation in the case of WordGroups and PermGroups). This is sometimes written f^t . Either program takes each element f in turn and forms the product $tf t^{-1}$ with every element t in the group. The resulting elements, g , are collected in a set together with f . It turns out that there are only a limited number of such sets – all elements in the group are in one and only one set, called a ‘conjugacy class’. Put another way, conjugation is an equivalence relation so that conjugation with any element t partitions the set of elements in G into disjoint conjugacy classes.

Conjugacy classes are important because conjugate elements within any one class share features peculiar to that class, such as having the same order. Under the Advanced Option of the programs, the character tables which the program later tries to calculate are an array of numbers which summarise and to some extent characterise the properties of the group. There is precisely one column of the character table of G for each conjugacy class.

Step 4): Calculate the derived subgroup (commutator subgroup) H_d .

A commutator element is a group element which can be made by the product (concatenation) of four elements in the pattern $g^{-1}f^{-1}gf$, or equivalently $fg^{-1}f^{-1}g$. It is a conjugate of g multiplied on the left by the inverse of g , often written $[f, g]$. To see its relation to commuting elements, observe that if f and g commute, so will f^{-1} and g^{-1} and hence $g^{-1}f^{-1}gf$ reduces to $g^{-1}f^{-1}fg = g^{-1}g = I$. So commutator elements are ones formed from non-commuting pairs. The fewer commuting elements there are in G, the more commutator elements are likely. The set of all commutators is not necessarily a group but, if it isn't, either program will calculate products of the commutators until

the additional elements do make it up to a group – called the ‘derived’ or ‘commutator’ subgroup, H_d . This is always a normal subgroup (that is, its right cosets are the same as its left cosets).

Step 5): Calculate the abelianisation factor group, G/H_d .

A subgroup which is invariant under the operation of forming conjugates is called a ‘normal’ subgroup. This means that, as sets, the normal subgroup and the conjugates of all its members are the same. This is equivalent to the left cosets being set-wise identical to the right cosets. Like the conjugacy classes, these cosets form a partition of G . Each of the cosets behaves like a *composite* element of a group called the ‘factor group’ G/N . The factor group of the derived (commutator) subgroup is called the ‘abelianisation factor group’ and it is always abelian (that is, every element commutes with every other element).

Either program finds the cosets of the derived subgroup, H_d . There are $|G|/|H_d|$ cosets each with $|H_d|$ elements, where $|..|$ denotes number of elements.

Step 6): Investigate subgroups and their cosets

Either program lists all the subgroups found so far, these being the centralisers of the various elements plus the centre Z and the derived/commutator subgroup. The list of subgroups will not necessarily contain all the subgroups of G . To help, WordGroups lists the powers of all elements, useful for spotting cyclic subgroups, and the products of pairs of order 2 elements whose product also has order 2 – this indicates a group of type $V = C_2 \times C_2$. You are offered the option to enter additional subgroups from the keyboard. You select one listed subgroup, H , at a time and the program examines the order of each element. It lists all elements which have the same order as H itself. Some subgroups have no such elements, but where one does, the multiples must run through all elements in the subgroup, so generating the whole of H . Thus H is cyclic.

If the Advanced option is selected, and for a *cyclic subgroup only*, the program will work out the character table and display it.

If the group is not cyclic, you will probably need to investigate it further so the program allows you to find multiples of selected elements and products of any two or three elements. By reference of known properties of smaller groups you should be able to identify the subgroup.

At this stage the program offers the option to calculate the cosets of a subgroup. It finds the left and right cosets, but leaves it to the user to decide whether, as sets, these are equal. If they are, the subgroup is normal, by definition. Note that any normal subgroup (not just the commutator subgroup) has the property that its cosets form a factor group, so you have the opportunity to explore these too.

An interesting exercise at this stage of a basic course is to draw the lattice diagram showing how all the subgroups fit within G , and identifying all the normal subgroups and their factor groups.

More Advanced Material: Group representations and characters

Steps 7 to 10 below calculate matrix representation and group characters so could be used to support study in a second level course. This analysis is carried out if you select ‘y’ to the Advanced Option which is offered about half way through either program.

Step 7): Sketch out the character table for G .

In order to explain group characters we would need to discuss group representations by matrices, and by transformations of vector spaces under group actions, but this would take too long so I refer you to textbooks and other web sites. Briefly, the elements of a group can be mimicked by a set of

appropriate invertible square matrices, with the group product represented by matrix multiplication. The trace (sum of diagonal elements) of the matrix corresponding to each group element is sufficient to capture key information about the matrix and hence the group element. It turns out that all elements in the same conjugacy class have the same trace (though not the same matrix). A group character is a list of the traces of the representing matrices, one for each conjugacy class. In general these will be complex numbers.

A group can be represented by many sets of matrices with arbitrarily large dimension. It turns out that the larger dimensioned matrices can be converted into a block diagonal form by a matrix similarity transformation. Each n -by- n diagonal block can then be identified with an n -by- n matrix which cannot itself be further split up. These are the ‘irreducible’ matrices representing the group. Their traces constitute the irreducible characters of the group. The character table is a square array of (generally) complex numbers, each row being one of the irreducible characters, and each column corresponding to one conjugacy class. The character table is a summary of the way the group’s actions can be represented by irreducible transformation matrices.

If a matrix representation involves a different matrix for each group element, so that the full symmetry properties of the group are manifest by the set of matrices, the representation and its corresponding character are said to be ‘faithful’. Not every group has a character which is both faithful and irreducible, but if one does exist it generally is at the bottom of the character table. Characters towards the top of the table may correspond to non-faithful representations. In a non-faithful representation some information about the group has been lost. A non-faithful representation is analogous to the shadow of an object cast on a wall – the shadow does represent the object, but only in a limited way. The first row in a character table is the ‘trivial representation’ in which every group element is represented by the identity. This is akin to an object being represented by a single point, so loss of information is almost total.

One significance of the abelianisation factor group is that it determines the linear (one-dimensional) characters in the character table of G . As a step to finding the linear characters of G we need the character table of the abelianisation factor group. The program calculates this table for simple cases.

The program prints out the skeleton of the character table of G , including these features:

- i) a header block labelling the conjugacy classes, naming a representative element for each class and summarising the order of each element and the number of elements in each class.
- ii) the likely values in the left hand column on the table for the identity class. The elements in this column have the property that the sum of their squares adds to $|G|$, and for some values of $|G|$ this is sufficient to determine uniquely what the squares must be.
- iii) all the linear characters. Linear characters correspond to representation matrices of dimension 1 and there are as many of them as cosets of the derived subgroup. Moreover there is a procedure calling ‘lifting’ by which the linear characters of G can be copied from the character table of the abelianisation factor group, calculated in Step 5.

The non-linear characters are left blank apart from the first column. Determining them can be a challenge and neither program attempts to do so directly. However each program does offer some help by reminding the user of the orthogonality properties of the character table, and by allowing further, usually reducible characters to be induced from the character tables of known subgroups. In the course of doing so some properties of subgroups are determined.

Both programs have a special facility to detect the direct (Cartesian) products of two cyclic groups and calculate the corresponding character table.

Step 8) Induce a matrix representation and character of G from a subgroup H.

'Induction' was discovered by Frobenius as a means of determining a character of group G from the character of a known subgroup H. The dimension (also called degree) of the induced character and associated matrix representation is $K = |G|/|H|$. If the character in H which is being induced has degree k, the induced matrix representing G has dimension kK , and so does the induced character. Clearly, in many cases kK will be a large integer and the induced representation will probably then be reducible.

The program calculates the induced character. In addition, where $k = 1$ (a linear character in H), it calculates the K-by-K induced matrices for the generators **a** and **b** of G in WordGroups, or for the first two seeded permutations in PermGroups. Induction of the matrices involves these steps

1. find the left cosets of H and a set of representative elements of these (a 'transversal'). Let the representative elements in the transversal be I, g_2, g_3, \dots, g_K , with g_2 representing the second coset, g_3 the third, etc..
2. calculate conjugate-like products of the form $g_i^{-1} a g_j$ to build up a K-by-K matrix which is intermediate to the calculation. $g_i^{-1} a g_j$ (after applying the relations of G) will give the element in the i^{th} row, j^{th} column of this intermediate matrix for element a .
3. examine each element in the intermediate matrix above and replace it by either a) its value in the character table of H if $(g_i^{-1} a g_j)$ is an element of H or ii) 0 if $(g_i^{-1} a g_j)$ is not in H.

The program prints the matrices for the generating elements **a** and **b** (or first two seeded permutations).. It also gives these in the formats suitable for input into the symbolic algebra packages Mathematica, Maxima/Macsyma and Reduce. Using one of these packages is the easiest way to multiply the induced matrices for **a** and **b** to generate matrices for the other group elements, and check that they have all the intended properties of the group. This set of matrices is then a representation of the group of dimension $K = |G|/|H|$.

The induced character is calculated whether or not the character in H is linear. Either program provides the option to induce other characters from the same subgroup, or examine other subgroups.

Step 9) Complete the character table and compose the lattice diagram of subgroups

This step is done by the user either by hand or using a symbolic algebra package. You can apply the orthogonality relations for the character table and solve the resulting simultaneous linear and quadratic equations to determine all the non-linear characters of G, and check them for consistency. You can run the program again as a calculator to investigate products of group elements in the search for other subgroups and draw up a lattice diagram to show which subgroups are themselves subgroups of others. By this stage you should have a pretty good understanding of the group.

Note on running the programs

I offer little apology for the crudeness of these computer program. They are both written in BBC Basic, originally devised in the 1970s for the BBC home and school micro-computer which became immensely popular in Britain. Though the BBC computer has long been superseded, its programming language survives in a simulation under Windows developed by R. T. Russell (web site <http://www.rtrussell.co.uk/>). This has the capacity to compile programs, and the compiled versions as '.exe' files are what I offer. They run as console programs with keyboard input when prompts appear, so they are interactive to this extent. They should run under Windows by double clicking on the program name.

Using Windows Explorer or otherwise, create a directory/folder for your output files such as C:\WG. When the program starts it offers the option of saving your output to file. You will need to enter the full path of the file such as C:\WG\run1. The program will add a 3-figure random second part to this name plus the file extension .rft. For example, C:\WG\run1_f8z.rft The added part of the name is to ensure that a new file is created at each run so that previous results are not overwritten. (The earlier version of the program would overwrite any existing file of the same name.) . The output to the screen has more detail than that to the file.

At the start of either program you are offered the option of having the screen scroll lock on or off. If it is on, press <SHIFT> to move to the next page. You cannot scroll backwards to review previous screens. You may find it useful, therefore, to run the program twice for each group because the output printed to file in the early part of the program contains information on group elements, their orders, etc. which is useful to have to hand while running the later analysis.

The output to the screen and printer with BBC Basic has limited formatting capability and I couldn't be bothered with making the printed tables and matrices align to look tidy. I suggest the following solution. If you run under Windows, go into C:\WG\ in Windows Explorer and open the .rft (rich text format) file in a word processor like Microsoft Word, save it as a .doc file and tidy the format to your satisfaction. You will need a fixed spacing font such as Courier or Prestige. For larger character tables landscape format will be needed.

About half way through each program you are offered the Advanced Option, to include group characters and representations. Students taking their first course in group theory should enter 'n' to skip this option, so as not to be over-faced with material you do not need to know.

Note on Permutation Cycle Notation

PermGroups uses the cycle notation for permutations, and the text in Part 2 regarding WordGroups also makes reference to permutations.

In cycle notation we start with N items placed in N initial positions labelled 1, 2, 3, ... N (or *a, b, c, ...* etc.). Cycle (12) means the item in position 1 moves in the sense \rightarrow to the place initially occupied by item 2, and item 2 moves to position 1, so items 1 and 2 swap places. It could be written $(1 \rightarrow 2 \rightarrow 1)$, but the convention is that the leading digit is not repeated. (12) is an example of a 2-cycle since there are two symbols in the bracket. (234) is a 3-cycle. It means that 2 moves \rightarrow to position 3, 3 to position 4, and 4 to position 2, with the further implication that item 1 stays where it is. For emphasis this may be written (1)(234). The identity is (1)(2)(3)...(N-1)(N), or indeed any subset of 1-cycles such as (1)(2)(3), or even simply (1). PermGroups uses (1) as the group identity.

Permutation cycles can be 'multiplied' as follows. Write two cycles next to each other and follow the convention that the cycle to the right is performed first.¹ For example, (23)(13) means perform (13) first, so that item 1 moves to position 3. Under the left hand cycle (23) the item at 3 moves to position 2, so overall 1 moves to 3 then to 2. Record this in a product as (12...). Next deal with the last item entered into the product, which is 2. Operation (13) does not affect 2, but under (23) item 2 moves to position 3. The product is now (123...). Finally under (13) item 3 moves to 1 and stays there. Thus the completed product is (123).

¹ Some authors perform the left operation first and PermGroups has the option to calculate either way.

An operation to swap precisely two elements, such as (12), is called a 'transposition'. Any permutation can be written as a sequence of transpositions: for instance, $(1342) = (13)(24)(23)$.

The result of carrying out a multiplication is to express the product permutation as a set of disjoint cycles; that is, cycles with no symbols in common. For example with $N = 6$ and evaluating right to left, $(1245)(256)(3612)(1546) = (16)(234)(5)$.

Cycle notation is particularly suited to showing conjugate elements because conjugates have the same structure of disjoint cycles.

In the program PermGroups cycles are entered without brackets and disjoint cycles are separated by any of the characters

' / \ * x . ,

Thus $(123)(2567)(15)$ could be entered as `123'2567'15` or `123\2567*15` etc.

So which orders of **a** and **b** do give finite groups? – or at least pass the tests for being closed under concatenation and have no duplicate inverses? Using a variant of the program I have searched for combinations of a^n and b^m which seem consistent and these are listed in Appendix A. All future examples will be chosen from this list.

2.1.2. More than one relation? The program allows three relations to be entered in addition to the orders of the two elements. To generate a finite group one relation of the form $ba = \dots$ is required. D4 and Q in §2.6 need two and S4 in §2.9 needs three. Three relations become needed when there are three or more generators **a**, **b**, **c**, ... , but WordGroups does not cater for these. If you enter two arbitrary relations, they will probably be inconsistent.

For example, suppose we select $a^3 = b^5 = I$, $b^2a = ab$, $ba^2 = a^2b$. Then
 $b^2a^2 = (b^2a)a = (ab)a = aba$ $a^3 = a(ba^2)a^2 = a(a^2b)a^2 = ba^2$
 which implies that $b^2 = b$ and hence $b = I$ in contradiction to $b^5 = I$.

2.2 Cyclic group C5

C5, the cyclic group of order 5 is also called Z_5 . Run WordGroups, input 'y' to save to file, and enter a full-path file name like C:\TGG\C5.

Enter order of **a** is 5 and of **b** is 5. There is 1 relation with LHS **b** and RHS **a**.

Effectively this relation means that **b** is identical with **a** so there is only one generator of order 5. Since 5 is prime, we expect a cyclic group and this is indeed what the program generates. It outputs the five elements as I, **a**, a^2 , a^3 , a^4 and their inverses. Satisfied of their consistency, the program proceeds as if these elements do form a group, G. Since $ba = ab$, all pairs of elements commute, so the group is said to be 'abelian'.

The first step in the analysis of the group is to find the order of each element. It states that there are

```
1 element of order 1 : 1
4 elements of order 5 : 2 3 4 5
```

where the number of an element refers to the listing at the beginning of the output. For example, element 3 is a^2 .

You are then offered the option of calculating products of elements, as in §2.1 above. Try it. Type 5 <RTN>, 4 <RTN> 1 <RTN> (or equivalently 5, 4, 1 <RTN>). It replies that the product is a^2 . What this means is that element 5 (a^4) concatenated with element 4 (a^3) then element 1 (I) gives a^7 which reduces mod 5 to a^2 . To close the product calculator enter '-1 <RTN><RTN> <RTN>' and move to the next stage.

The program now finds all the elements which commute with a chosen element. Recall that f and g commute when $fg = gf$. It takes each element g in turn and determines its order, then tests g with all elements in the group in turn to see if they commute with g . The word 'central' is used in association with commuting elements, and the set of all elements which commute with g is called the 'centraliser' of g . This set always forms a subgroup of G. As an example, for element 3 the output reads:

```
Element 3 = a^2 has order 5.
Its centraliser subgroup has 5 members: a^2 itself +
I a a^3 a^4
Corresponding element numbers are
1 2 4 5
```

You will see this same set of five centraliser elements appearing against every group element. This is the nature of a cyclic group. At the end of the list of centralisers the program identifies the centre Z as the whole group. The centre Z of a group is the set (which is also a normal subgroup) which commutes with *every* element in G . So here Z is the whole group.

The next stage of the program works out the conjugacy classes. Recall that g_2 is conjugate to g_1 if there is at least one other group element x and its inverse x^{-1} such that $g_2 = x^{-1}g_1x$. This structure, with g_1 sandwiched between another element and its inverse, is very like the structure of similarity transformations between linear operators g_1 and g_2 and between similar matrices g_1 and g_2 .

WordGroups takes each element g_1 in turn and combines it with every other element g_2 , recording for each combination the resulting element (after reduction by the defined orders of \mathbf{a} and \mathbf{b} , and by the specified relations $\mathbf{ba} = \dots$, etc.). Elements which are conjugate to one another form a conjugacy class. The program collates these classes, first putting them in order of the element which created them, and then in order of increasing order of element. (Sorry about the two meanings of 'order' there!) Because in a cyclic group each element commutes with every other, each element is conjugate only to itself. There are therefore five conjugacy classes.

At this point you have the choice of exiting or continuing with more advanced topics.

When you select the advanced option, the program calculates the commutator elements of G . These are elements which can be 'factored' into the form $g^{-1}h^{-1}gh$. For group C_5 there is only one such element – the identity I . In some groups the set of commutators is not necessarily closed so the program calculates powers and multiples of the elements to extend the set to a full subgroup. It reports whether elements have been added. This subgroup made from commutators is the 'commutator' or 'derived' subgroup. For C_5 it is the trivial group I . The program calculates the cosets of the derived subgroup. These will be important for more complicated groups, but for C_5 they are just the other elements of the group itself.

The program now comes to a section headed 'Structure of subgroups and induced characters'. It lists the subgroups found so far from the centralisers of the elements and the asks if you wish to enter another subgroup from the keyboard. Since the only subgroups of C_5 are $\{I\}$ and C_5 itself, there is little point, so type 'n'. It then asks if you wish to examine a subgroup. Enter 'n' again. Decline all further options. That concludes our basic-level investigation of C_5 .

Just a note about C_5 expressed in permutation cycle notation. Suppose \mathbf{a} is the cycle (12345). Then $\mathbf{a}^2 = (13524)$, $\mathbf{a}^3 = (14253)$, $\mathbf{a}^4 = (15432)$ and $\mathbf{a}^5 = (1)(2)(3)(4)(5) = I$.

Advanced option: When you are asked whether you wish to include the more advanced material on group representations, reply 'y'.

The goal of the analysis is to produce the group character table and identify most of its subgroups. The program prints the regular character based on C_5 being a subgroup of S_5 , the symmetric group of permutations of 5 items. It also refers to a property of character tables: namely that there are as many irreducible characters as there are conjugacy classes, so a character table is a square array.

When the cosets of the derived subgroup are printed, you are asked 'do the first elements of the cosets form a cyclic subgroup of order 5?'. They clearly do, so type 'y'. It asks for the letter used to generate this cyclic group, so type 'a'. The program calculates the character table of the cyclic group. It uses the notation e to stand for $\exp(2\pi i/5)$, e^* for $\exp(-2\pi i/5)$, e^2 for $e^2 = e^2$, etc.

If the program were just designed for cyclic groups, it could end there. In the case of C5 all the five characters are linear (one-dimensional). Over page is the complete character table for C5. The five conjugacy classes are labelled K1 to K5. This table means that C5 can be represented in five distinct ways by a set of one-dimensional matrices. These are the five irreducible representations of C5; ‘irreducible’ because a 1-by-1 matrix cannot be further decomposed into smaller matrices.

The four representations containing e are all faithful, meaning that they fully embody all the symmetry properties of the group, with no information being lost. Thus, in row 2, a is represented by the 1-D matrix $[e] = [\exp(2\pi i/5)]$, so a^2 corresponds to $[e^2]$, a^3 to $[e^3] = [e^{2*}]$, a^4 to $[e^4] = [e^*]$. The order of each of these is 5. In row 3 a is represented by the matrix $[e^2]$, making a^2 correspond to $[e^4] = [e^*]$, and so on.

Group C5					
Class label	K1	K2	K3	K4	K5
Element id no.	1	2	3	4	5
Element	I	a	a ²	a ³	a ⁴
Number in class	1	1	1	1	1
Order of element	1	5	5	5	5
Order centraliser	5	5	5	5	5
<hr/>					
Trivial char.	1	1	1	1	1
	1	e	e ²	e ^{2*}	e [*]
	1	e ²	e [*]	e	e ^{2*}
	1	e ^{2*}	e	e [*]	e ²
	1	e [*]	e ^{2*}	e ²	e

The first character (first row) is not faithful. It corresponds to the trivial representation in which every element is represented by the matrix $[1]$. This means that every group element is mapped to the identity matrix. So the kernel of the homomorphism which does this is the whole of G.

2.3 C12 and the abelian direct product C4×C3

2.3.1. C12

Generate the cyclic group C12 in the same way as you generated C5. Enter order of a is 12, order of b is 12, number of relations is 1, LHS is b , RHS is a and follow the steps as for C5 above.

Note that in checking whether the elements do indeed form a group, the program calculates all pairwise products of elements. This can cause a noticeable delay before further output. For groups with more than 16 elements you have the option of not performing this check, though I recommend that you do perform it once for every new group.

The feature to notice with C12 is those elements which have order 12. It is not all of them, but only the four for which the order of the element is coprime with 12: that is, elements a^1 , a^5 , a^7 and a^{11} . There are two elements with order 3 (a^4 , a^8) and each of these generates the abelian subgroup with three members $\{I, a^4, a^8\}$.

When you come to the ‘structure of subgroups and induced characters’ part of the program, say ‘y’ to entering a new subgroup from the keyboard. It asks for the order of your subgroup. We will first

enter some incorrect data to show that the program performs checks on the validity of the proposed subgroup.

- a) Type 8. The program responds that (by Lagrange's theorem) the order of the subgroup must divide the order of the group.
- b) Type order is 3, and enter the three elements by reference as 1 <RTN>, 4 <RTN>, 10 <RTN>. It again detects an error in that the square of element 4 is not present in the list. The missing element is 7. So this set is not closed and cannot constitute a subgroup.
- c) Type 4 elements, then 1<RTN>, 4 <RTN>, 7 <RTN>, 10 <RTN>. This is now a valid subgroup. For the time being we will not examine it further. So say 'n' to 'do you want to examine one of these subgroup?'
- d) Enter 'y' to calculating products and check that elements $5 \times 9 = I$. You are obliged to enter 3 elements so make one the identity by entering 1 <RTN>, 5 <RTN>, 9 <RTN>.
- e) You are again given the option to enter a subgroup from the keyboard. Type 3 elements, and then enter 1 <RTN>, 5 <RTN>, 9 <RTN>. These also form a subgroup and the program proceeds. The two new subgroups appear as numbers 4 and 5 in the list.

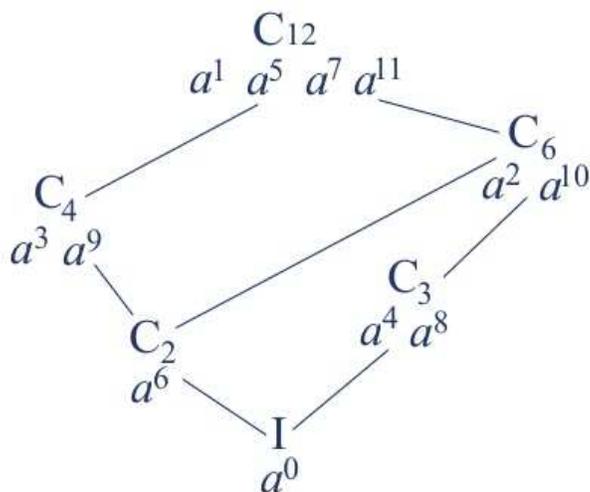
Select subgroup number 5 and accept the offer to calculate its cosets. The program reports that the left cosets are

$$\begin{aligned} \{ 1 \ 5 \ 9 \} & : \text{ <- this is subgroup No. 5 left multiplied by I} \\ \{ 2 \ 6 \ 10 \} & : \text{ <- this is subgroup No. 5 left multiplied by a} \\ \{ 3 \ 7 \ 11 \} & \\ \{ 4 \ 8 \ 12 \} & \end{aligned}$$

and the right cosets are identical. By definition, then, this subgroup is normal. Its four cosets are the four composite elements of a factor group C_{12}/C_3 . The program lists elements $\{1, 2, 3, 4\} = \{ I, a, a^2, a^3 \}$ as a set representing each of the four cosets. This representative set is called a 'transversal'.

You now have enough information to sketch the diagram below which shows how the elements of C_{12} divide into several subgroups. You read this from the bottom, accumulating the elements noted along the lines. Thus I and a^6 from C_2 , the cyclic group on 2 elements. Adding a^3 and a^9 produces the cyclic group C_4 generated by a^3 . C_6 is generated by a^2 .

In contrast to C_{12} , C_5 and other cyclic groups of prime order have no non-trivial subgroups. The prime cyclic groups are a species of 'simple' group – a group with no non-trivial *normal* subgroups and hence no factor groups. Interest in group theory has focused on the simple groups (simple in the technical sense of the word only!) as the primitive building blocks from which more complicated groups are composed — the 'prime numbers' of group theory.



Advanced option : It is possible to induce a reducible character from each of the subgroups, but I suggest you come back to that exercise once you have completed the more interesting exercises on other groups below. Decline all further options and end the run.

2.2.2 $C_4 \times C_3$

A ‘direct product’ group is one in which the elements are the Cartesian products of the elements in each constituent group. All the constituent groups become normal subgroups in the product group. Here we have two cyclic constituent groups. The majority of groups are direct or semi-direct products of two or more constituent groups, for example $C_2 \times C_3 \times C_4$, $D_6 \times A_4$, $Q \times S_4$, etc.

If we were to build $C_4 \times C_3$ from permutation cycles (say using PermGroups), the two single generating elements would be $\mathbf{a} = (1234)$, $\mathbf{b} = (abc)$. This makes it clear that a direct product is the *disjoint* product of two permutations. Disjoint products commute, and so the direct product group is abelian.

Run WordGroups, enter a suitable file name, give generator \mathbf{a} order 4 and generator \mathbf{b} order 3. Say there is 1 relation with LHS ‘ba’ and RHS ‘ab’. It is not difficult to show that the commuting of \mathbf{a} and \mathbf{b} means that $\mathbf{b}^j \cdot \mathbf{a}^k = \mathbf{a}^k \cdot \mathbf{b}^j$ for all powers j and k . Hence the group is abelian.

The program finds 12 elements, calculates their orders and centralisers, the conjugacy classes, and the derived subgroup as with C_5 and C_{12} . Note that the commutators are all the group elements, and that there are 12 conjugacy classes, each with only one element. This is the behaviour we would expect of a cyclic group. There is an important example of isomorphism revealed here. Look at the orders of the elements in $C_4 \times C_3$. The program prints

```
4 elements of order 12 : 7 8 11 12
```

If there are elements of order 12, the group of 12 elements *must* be cyclic. Let’s look at this further.

When the program comes to listing the subgroups found, you will see that it lists two identical rows of all 12 elements, but the subgroups of order 4 and 3 have not been detected because they do not correspond to centralisers of any elements. Enter ‘n’ when asked if you wish to enter a new subgroup from the keyboard: we will do that a little later. Enter ‘y’ to ‘Do you wish to examine one of these subgroups?’ Type ‘1’ to select the Subgroup No.1. The program re-examines the orders of all elements and prints those with order 12, commenting that the group must be cyclic. You can now explore the powers of any element by typing its number. For example, element 7 gives rise to the output

```
Multiples of element 7 which is ab
Power 2 is a^2 b^2 (element 10)
Power 3 is a^3 (4)
Power 4 is b (5)
Power 5 is a b^2 (8)
Power 6 is a^2 (3)
Power 7 is a^3 b (11)
Power 8 is b^2 (6)
Power 9 is a (2)
Power 10 is a^2 b (9)
Power 11 is a^3 b^2 (12)
Power 12 is I (1)
```

confirming that \mathbf{ab} does indeed generate the whole group. So $C_4 \times C_3$ must be isomorphic to C_{12} , which we examined in §2.3.1. Type ‘-1’ to decline further powers, and ‘n’ to calculating products of elements.

You get a second chance to enter another subgroup from the keyboard. Select ‘y’. One subgroup is clearly the cyclic group on \mathbf{a} with four elements. Enter the order of the subgroup as 4, then enter

element numbers 1 <RTN> 2 <RTN> 3 <RTN> 4 <RTN>. When asked if you wish to examine one of the subgroups, type 'y' and the subgroups will be relisted, with the new subgroup added as Subgroup No. 4.

Move down to the option 'do you wish to calculate the cosets?' and say 'y'. The program prints the left and right cosets and you will see that these are identical. Hence the subgroup C4 is normal in C12. Moreover the first elements in the three cosets are $\{I, b, b^2\}$, which is the cyclic group C3 generated by **b**. You can add the subgroup on **b** with elements $\{1, 5, 6\}$ and find that it also is normal. This is a characteristic of a direct product $G = H \times K$; both the generating subgroups H and K are normal in G, and the factor group G/H is isomorphic to K, and *vice versa*.

Decline further options and exit the program. But here's a challenge: convince yourself that the centre Z of a direct product is the direct product of the centres of the factor groups which form it..

Advanced Option: WordGroups recognises the input conditions as generating a direct product of cyclic groups and goes directly to printing the character table for this group. It first prints the character table for the first generator **a**, which in this case is C4. Then it prints the table for C3, since $b^3 = I$. The character table of $C4 \times C3$ is made of 3^2 blocks each of 4^2 elements. Effectively the C3 table is modified by having every character element in it replaced by the whole C4 table multiplied by the value of the replaced C3 element. (I apologise for the frayed formatting at the right hand side of this character table; you will have to add spaces to improve the column layout.)

If you were to input the group as $C3 \times C4$, by setting the order of **a** as 3 and order of **b** as 4, with $ba = ab$ still, you would get an equivalent character table with rows and columns in different orders. For example, column 3 in $C4 \times C3$ is for the conjugacy class of a^2 and is equivalent to column 7 in $C3 \times C4$. In fact $C3 \times C4$ is isomorphic to $C4 \times C3$; they are essentially the same group with different labelling of the elements.

The program next will calculate the character table of the cyclic subgroup and you are asked to select one of the elements which will be its generator. Here elements 7, 8, 11 and 12 should generate C12, so type '7'. (You could try the other elements too.) It calculates the character table of the cyclic group generated by powers of element 7. The elements corresponding to these powers are listed in the header to the table. Thus element 3 = a^2 is $(ab)^6$ and its conjugacy class is in column 7 of the C12 table.

You can see the isomorphism of C12 and $C4 \times C3$ by comparing their character tables. If the equivalence is not obvious, bear in mind that $-i.E = -i.exp(2\pi i/3) = exp(2\pi i/12) = 'c'$, and that $'E' = exp(2\pi i/3) = exp(4*2\pi i/12) = 'c^4'$.

If this is your first reading of these notes, decline the option to induce a representation. You can come back to that later when we have examined induction in some more interesting groups.

2.4 Abelian direct products $C2 \times C2$, $C3 \times C3$ and $C4 \times C4$

The aim in this exercise is to see that $C2 \times C2$ is not isomorphic to C4, $C3 \times C3$ is not isomorphic to C9, and $C4 \times C4$ is not isomorphic to C16. We will also look at finding the subgroups in these, since neither WordGroups nor PermGroups will find all these subgroups automatically.

2.4.1 $C2 \times C2$

This important direct product is called the Klein 4-group and denoted V after the German 'Viergruppe'. Run WordGroups and generate $C2 \times C2$ by entering **a** with order 2, **b** with order 2, 1

relation with LHS **ba** and RHS **ab**. The elements a, b, ab all have order 2 so no element can generate the whole group. It is therefore not cyclic, *i.e.* it is not isomorphic to C4.

Select 'y' to the advanced option. Four conjugacy classes are found and the program recognises the group as C2×C2 and prints its character table as

I	a	b	ab
1	1	1	1
1	-1	1	-1
1	1	-1	-1
1	-1	-1	1

The only subgroup found automatically is the whole group. When promoted, accept the offer to enter a subgroup from the keyboard. Enter order 2, and element numbers 1 <RTN> 2 <RTN>. You have proposed that {I, a} from a subgroup. The program checks products and accepts this as a valid subgroup. A list of all subgroups found so far is displayed in which the new subgroup is number 3. Select to examine this further.

WordGroups recognises this as C2 and prints its character table. The one non-trivial character corresponds to I being represented by the one-dimensional matrix [1] and a by [-1]. To find a pair of 2-D matrices which represent the parent C2×C2 group, select to induce a character, and then row 2 (the first non-trivial character of C2×C2) in the above character table. The program generates

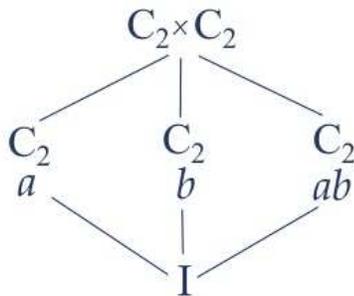
Matrix representation of 'a' is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	Matrix reprn. of 'b' is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
---	---

and prints these in formats for three common symbolic maths packages. These can be used to calculate the matrices

$$ab = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The traces of these matrices for I, a, b, ab respectively are 2, -2, 0, 0 and this is the induced character.

You can enter by hand two other subgroups: {I, b} and {I, ab} and explore them similarly. The subgroup structure diagram is



The elements added to those below to form the subgroup are given.

2.4.2 C3×C3

Run WordGroups again and generate C3×C3 by entering **a** with order 3, **b** with order 3, 1 relation with LHS **ba**, RHS **ab**.

A group of 9 elements is generated. Note the orders of the elements; there are 8 elements of order 3 and none of order 9. This group is *not* cyclic even though it is abelian. C3×C3 cannot be isomorphic to C9. This again illustrates a general principle; that C_n×C_m is isomorphic to C_{nm} only

when n and m are coprime, for only then can there be at least one Cartesian product element whose multiples scan through all possible suffices nm and so generate the whole cyclic group. This is seen even more vividly in cycle notation with $\mathbf{a} = (123)$, $\mathbf{b} = (abc)$.

Select the advanced option. The program again invites you to investigate subgroups. $C_3 \times C_3$ does have four subgroups of order 3, e.g. $\{I, \mathbf{a}, \mathbf{a}^2\}$, $\{I, \mathbf{ab}, \mathbf{a}^2\mathbf{b}^2\}$, but the program has not found these because they are not centralisers of any element. This hints at the difficulty of finding all subgroups, especially when a group has a highly composite order. Clues are in the orders of the elements. After the character table has been printed, select to enter a subgroup from the keyboard. Enter order 3 and element numbers 1 <RTN> 4 <RTN> 5 <RTN>. These form the cyclic subgroup C_3 on $\{I, \mathbf{a}, \mathbf{a}^2\}$. By inducing a character of $C_3 \times C_3$ as in §2.4.1 you can obtain the matrix representations

Matrix representation of 'a' is

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

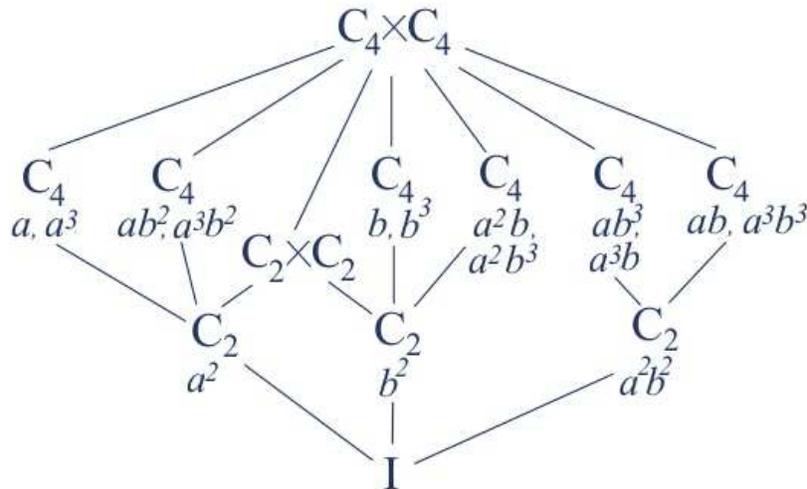
Matrix reprn. of 'b' is

$$\begin{pmatrix} c^* & 0 & 0 \\ 0 & c^* & 0 \\ 0 & 0 & c^* \end{pmatrix}$$

where $c^* = \exp(-2\pi i/3)$. You are also invited to draw the subgroup lattice diagram.

2.4.3 $C_4 \times C_4$

Run WordGroups again and generate $C_4 \times C_4$ by entering \mathbf{a} with order 4, \mathbf{b} with order 4, 1 relation with $\mathbf{ba} = \mathbf{ab}$. Use the listed powers of elements and the direct products of elements of order 2, and show that the subgroup structure diagram is



2.5 A symmetric group $S_3 =$ a dihedral group D_3

The symmetric group S_3 encompasses all the permutations of three items. These could correspond physically to the corners of an equilateral triangle subject to rotations and reflections. A regular plane polygon with n sides has the symmetry group D_n , where D stands for dihedral.² Hence S_3 and D_3 are the same group of $3! = 6$ elements. This is the smallest non-abelian group.

Generate the group by entering order of \mathbf{a} is 3, order of \mathbf{b} is 2, 1 relation with LHS \mathbf{ba} and RHS \mathbf{aab} . The program generates the six elements $\{I = \mathbf{a}^3 = \mathbf{b}^2, \mathbf{a}, \mathbf{a}^2, \mathbf{b}, \mathbf{ab}, \mathbf{a}^2\mathbf{b}\}$.

PermGroups will readily generate S_3 from permutations. These correspond to letting $\mathbf{a} = (123)$,

² There is an alternative nomenclature for dihedral groups in which the regular n -polygon has group D_{2n} with $2n$ elements.

$\mathbf{b} = (12)$. Then $\mathbf{a}^3 = \mathbf{b}^2 = (1)(2)(3) = \mathbf{I}$, $\mathbf{a}^2 = (123)(123) = (132)$, $\mathbf{ab} = (123)(12) = (13)$, and $\mathbf{a}^2\mathbf{b} = (123)(13) = (23)$. $\mathbf{ba} = (12)(123) = (23) = \mathbf{a}^2\mathbf{b}$.

There are three conjugacy classes:

Class 1 has 1 elements of order 1:	1	(I)
Class 3 has 3 elements of order 2:	4 5 6	(b, ab, a ² b)
Class 2 has 2 elements of order 3:	2 3	(a, a ²)

The three commutators are {I, a, a²}. The two cosets of the derived subgroup are represented by I and b, which form a cyclic group of order 2. This is the abelianisation factor group.

If you decline the Advanced Option, you move to the ‘Structure of subgroups’ section. The list of subgroups includes these centralisers :

Subgroup No. 1 of 3 elements	: 1 2 3
Subgroup No. 2 of 2 elements	: 1 4
Subgroup No. 3 of 2 elements	: 1 5
Subgroup No. 4 of 2 elements	: 1 6

Say ‘n’ to entering another subgroup from the keyboard, and ‘y’ to ‘do you wish to examine one of these subgroups?’. Select No. 1. The program states that it is cyclic. Look at its cosets. The two right and two left cosets are identical, so C3 is normal in S3.

When it asks ‘look at another subgroup?’ enter ‘y’ and select subgroups No. 2, 3 and 4 in turn. For subgroup No. 2 the output reads

Left cosets of subgroup No. 2 are :	Right cosets of subgroup No. 2 are :
{ 1 4 }	{ 1 4 }
{ 2 5 }	{ 2 6 }
{ 3 6 }	{ 3 5 }

So this is not normal, even though the left and right coset representatives are both {I a a²}. In §2.7 we will look at semi-direct products of cyclic groups. In the meantime just note that S3 is the semi-direct product of C3 and C2, written C3 \rtimes C2.

Advanced Option: When the program lists the cosets of the derived subgroup, enter ‘y’ to confirm that their first elements do form this cyclic group, and ‘b’ as the generating letter. The program prints the character table of C2 with b as generator, and the framework of the character table of S3. It completes the two linear characters by ‘lifting’ them from abelianisation factor group C2.

Basically ‘lifting’ means that all elements of S3 which are in the same composite element of the abelianisation factor group C2 (cosets of the derived subgroup) share the same character value. So the procedure for lifting a character from subgroup H to main group G is as follows:

- Look along the header of the G character table (here S3) and note the element which represents the conjugacy class for each column in turn.
- Find which coset of the derived subgroup this representative element occurs in
- Find the element in that coset which appears in the header of the table for the subgroup H (here C2)
- Choose a character (i.e. a row) in the table of H and find the character value for the element in question
- Copy that character value into the selected column of the table for G. Each character row in H induces one character row in G.

In this simple cases a is in the same coset as I and so has the same lifted character value: namely 1.

At this stage the program leaves two values of the S3 character table empty; these are in the one non-linear character. In general the biggest challenge in working out character tables is to find the non-linear characters. The first column has been completed by using the fact that the sum of the

squares of values in the first column equals the order of the group $|G|$. There is only one set of three positive integers whose squares add to 6, namely $1^2 + 1^2 + 2^2$. This is one of the orthogonality relations on the rows and columns of any character table.

With some groups a considerable amount of detective work using these orthogonality relations is needed to find the rest of the non-linear characters. There is one very useful technique, however, which can be used to ‘induce’ a character of G from the known character of a subgroup H . Induction was developed by Frobenius. To use this you must first identify a subgroup, so enter ‘n’ to ‘do you wish to enter from keyboard another subgroup?’ and enter ‘y’ to the request ‘do you wish to examine a subgroup?’.

Select subgroup No. 1 with the three elements $\{I, a, a^2\}$. The program replies that elements 2 and 3 have order 3 and the subgroup is cyclic. To check, type 2 to evaluate powers of element number 2. Say ‘n’ to calculating products of elements, and enter 2 as the element to generate the character table of this cyclic subgroup. It prints the table of C_3 using the notation $c = \exp(2\pi i/3)$. The induced character of G will be built from this.

Say ‘y’ to constructing an induced representation. The program lists the left cosets of the subgroup $H = C_3$ and picks out the first element in each coset as representative of that coset. (This set of coset representatives is called a ‘transversal’ of the factor group.) You are now asked to select a character from the C_3 table to induce up to G . Characters are numbered by row. Row 1 is just the trivial character and usually will not give the most useful result, so choose row ‘2’ and confirm your choice of $\{1 \ c \ c^*\}$. The program displays to the screen some intermediate steps in the induction procedure, but we focus on the result which is the two 2-D matrices for \mathbf{a} and \mathbf{b} and the induced character. These are given as

<p>Matrix representation of 'a' is</p> $\begin{bmatrix} c & 0 \\ 0 & c^* \end{bmatrix}$	<p>Matrix reprn. of 'b' is</p> $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
<p>Induced character is</p> <p>2 0 $c+c^*$ where $c = \exp(2\pi i/3)$.</p>	

The meaning is as follows. The six elements of $S_3 = D_3$ can be represented by six 2-by-2 matrices formed from the two above matrices for \mathbf{a} and \mathbf{b} plus their matrix products \mathbf{a}^2 , \mathbf{ab} and $\mathbf{a}^2\mathbf{b}$, plus the identity matrix \mathbf{I}

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{a}^2 = \begin{bmatrix} c^* & 0 \\ 0 & c \end{bmatrix} \quad \mathbf{ab} = \begin{bmatrix} 0 & c \\ c^* & 0 \end{bmatrix} \quad \mathbf{a}^2\mathbf{b} = \begin{bmatrix} 0 & c^* \\ c & 0 \end{bmatrix}$$

All of these matrices are different so this is a faithful representation – it captures all the properties of the group S_3 . Note that the matrices are complex and orthogonal (unitary). It is worth checking that $\mathbf{a}^3 = \mathbf{b}^2 = \mathbf{I}$ and that $\mathbf{ba} = \mathbf{a}^2\mathbf{b}$. For this a computer symbolic algebra package is most convenient, so the program outputs the \mathbf{a} and \mathbf{b} matrices in formats suitable to paste into Reduce, Maxima/Macsyma or Mathematica.

A character is, by definition, composed of the traces of the matrices in each conjugacy class. There are three classes, as identified above. \mathbf{I} is in a class by itself and the trace of the \mathbf{I} matrix is $1+1=2$. \mathbf{a} and \mathbf{a}^2 constitute another conjugacy class and the trace of their matrices is $c+c^* = -1$. The other three elements, containing \mathbf{b} , form the third class and for each the trace of its matrix is 0. This illustrates the general rule that the character value, being equal to the trace, is the same for all elements in one conjugacy class. The induced character is therefore $(2 \ 0 \ -1)$.

Let us look in more detail at the induction process. (Refer to Step 8 in Part 1.) An intermediate stage of the induction procedure requires count to be kept of the number of times an element of the form $(g_2^{-1} f g_3)$ is a member of H, where $\{I, g_2, \dots, g_k\}$ are a transversal representing the factor group G/H. The program lists these counts in an array, one column for each conjugacy class of G and one row for each element in H as follows:

Class label	K_1	K_3	K_2
Element id no.	1	4	2
Element	I	b	a
1	6	0	0
c	0	0	3
c*	0	0	3

For example, in the conjugacy class of G containing element 2 = a, a^2 was formed three times from products of the form $g_2^{-1} f g_3$. In H, a^2 has character value c^* (left most column). It is quite easy to work out the induced character in G from this array. Simply multiply the number of occurrences by the character value displayed to the left, sum down the column and divide by the order of h, |H|. The conjugacy class of a has induced value $(3c + 3c^*)/|H| = c+c^* = -1$, on substituting for c, in agreement with that found above from the trace of the matrices.

This completes the character table, which we give for completeness:

Group S3 = D3

Class label	K_1	K_3	K_2
Element id no.	1	4	2
Repr. Element	I	b	a
Number in class	1	3	2
Order of element	1	2	3
Order of centraliser	6	2	3
Trivial char.	1	1	1
	1	-1	1
	2	0	-1

WordGroups offers the option to induce another character from the same subgroup. Type 'y' and select character '3' from the cyclic subgroup. Note that the induced matrix representation is the complex conjugate of that from row 2 of the subgroup table. This leads to a general rule: if X is a character containing complex numbers, then X^* is also a character. (In this case the character is real even though the matrix representation is complex.)

Next say 'y' to looking at another subgroup. This time we will chose subgroup number 2, which has the elements {I, b}. Element 4 has order 2, the order of the cyclic subgroup, so say 'y' to the program calculating the C2 character table, and induce a character from this using row 2 which is {1 -1}. The program now induces a matrix representation of S3 of dimension $6/2 = 3$. It outputs

Matrix representation of 'a' is	Matrix reprn. of 'b' is
0 0 1	-1 0 0
1 0 0	0 0 -1
0 1 0	0 -1 0
Induced character is	
3 -1 0 .	

Again you can use a computer algebra program to check that these matrices, when combined by ordinary matrix multiplication, generate a group of 6 elements with all the properties of S3. The trace of the **a** matrix is 0 and that of the **b** matrix is -1. Clearly the identity matrix is 3-by-3 and have trace 3. This explains the induced character.

Why is this 3-D character not in the character table? The reason is that it is not an irreducible character. Rather, it is the sum of {1 -1 1} and {2 0 -1}, each of which is irreducible. The character table is made only of irreducible characters. In an intuitive sense, a reducible character is larger and more complicated than is necessary to capture all the properties of the group. The faithful irreducible representations, {2 0 -1} in this case, are exactly the right size to capture all information in a group with nothing to spare.

The fact that {3 -1 0} is a sum of the traces of a 1-D and a 2-D matrix is a consequence of a celebrated theorem due to Heinrich Masche. He proved that all representations of a group, whether by matrices or their corresponding linear transformations, are either irreducible or the sum of irreducible representations. By 'sum' here we mean a direct sum of subspaces; each and every matrix in a group can be put in the form of a block diagonal matrix by a suitable transformation of co-ordinate basis, and this one transformation is the same for every group element. Note that the 3-D matrix for b above is already in block diagonal form

$$\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \quad \text{where } P = [-1] \text{ and } Q = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

We can illustrate this conversion to block diagonal form by a change of basis, as follows. A change of basis of matrix M from basis B₁ to B₂ is effected by a similarity transformation involving a non-singular matrix T:

$$M|_{B_2} = T^{-1} M|_{B_1} T.$$

One matrix T which will produce a block diagonal structure for all matrices representing group G is

$$T = \begin{bmatrix} & -2 & 1 &] \\ [1 & --- & --- &] \\ [& 3 & 3 &] \\ [& & &] \\ [& 1 & -2 &] \\ [1 & --- & --- &] \\ [& 3 & 3 &] \\ [& & &] \\ [& 1 & 1 &] \\ [1 & --- & --- &] \\ [& 3 & 3 &] \end{bmatrix} \quad \text{which has inverse } 1/T = \begin{bmatrix} [1 & 1 & 1 &] \\ [--- & --- & --- &] \\ [3 & 3 & 3 &] \\ [& & &] \\ [-1 & 0 & 1 &] \\ [& & &] \\ [0 & -1 & 1 &] \end{bmatrix}$$

With this, matrix A transforms as $1/T * A * T =$
 $\begin{bmatrix} [1 & 0 & 0 &] \\ [& & &] \\ [0 & 0 & -1 &] \\ [& & &] \\ [0 & 1 & -1 &] \end{bmatrix}$ which is in (1x1 + 2x2) block diagonal form

It has trace = 1 + (-1) = 0

The matrices for the other elements of S3 transform as follows:

$$1/T * B * T = \begin{bmatrix} [-1 & 0 & 0 &] \\ [& & &] \\ [0 & -1 & 1 &] \\ [& & &] \\ [0 & 0 & 1 &] \end{bmatrix} \quad \text{with trace} = -1 + (0) = -1$$

$$1/T^*A^*B^*T = \begin{bmatrix} -1 & 0 & 0 \\ & & \\ 0 & 0 & -1 \\ & & \\ 0 & -1 & 0 \end{bmatrix} \quad \text{trace} = -1 + (0) = -1.$$

$$1/T^*A^*A^*B^*T = \begin{bmatrix} -1 & 0 & 0 \\ & & \\ 0 & 1 & 0 \\ & & \\ 0 & 1 & -1 \end{bmatrix} \quad \text{trace} = -1 + (0) = -1.$$

$$1/T^*A^*A^*T = \begin{bmatrix} 1 & 0 & 0 \\ & & \\ 0 & -1 & 1 \\ & & \\ 0 & -1 & 0 \end{bmatrix} \quad \text{trace} = 1 + (-1) = 0$$

The trace of a matrix remains unchanged under a similarity transformation so the group character, being the traces for each conjugacy class, is still $\{3 \ -1 \ 0\}$. The upper left 1×1 block in the matrices gives the character $\{1 \ -1 \ 1\}$ whilst the 2×2 block gives the character $\{2 \ 0 \ -1\}$.

Notice that although this is in a block diagonal form, the matrix for **a** is

$$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \quad \text{involving only real numbers, and not the complex representation}$$

$$\begin{bmatrix} c^* & 0 \\ 0 & c \end{bmatrix} \quad \text{where } c = \exp(2\pi i/3) \text{ as was found from the induction on } C_3.$$

Similarly the matrix for **b** is

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and not} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{as found by the } C_3 \text{ induction.}$$

This is because an irreducible representation can have many matrix representations, related to each other by similarity transformations. They all do have the same trace, however, (a key property of similar matrices) and indeed this is one reason why the traces of the matrices, in a character table, are such a powerful summary of many group characteristics.

2.6 Another dihedral group, D4

D4 is the full symmetry group of a square. It has a four-fold rotation axes and four mirror lines, so we expect a cyclic subgroup of order 4 plus four subgroups of order 2.

Enter **a** with order 4, **b** with order 2, one relation with LHS **ba** and RHS **aaab**. **aaa** means **a**³, which is the inverse of **a**.

All the dihedral groups D_n have generators **a**ⁿ = I, **b**² = I, **ba** = **a**⁻¹ **b**. The **b** generates the reflection symmetry and the **a** the *n*-fold rotational symmetry.

The program generates this set of elements

No.	Element	Inverse	Inverse No.
1	I	I	1

2	a	a ³	4
3	a ²	a ²	3
4	a ³	a	2
5	b	b	5
6	ab	ab	6
7	a ² b	a ² b	7
8	a ³ b	a ³ b	8

and reports the five conjugacy classes, sorted in ascending order of element, as:

```

Class 1 has 1 elements of order 1:      1          order of centraliser is 8
Class 3 has 1 elements of order 2:      3          order of centraliser is 8
Class 4 has 2 elements of order 2:      5  7      order of centraliser is 4
Class 5 has 2 elements of order 2:      6  8      order of centraliser is 4
Class 2 has 2 elements of order 4:      2  4      order of centraliser is 4

```

where I have added on the right the order of the centraliser. (All elements in the same conjugacy class have the same order of centraliser.) Note that

$$(\text{number of elements in conjugacy class}) \times (\text{order of centraliser for class}) = |G|.$$

This is true for all groups and is an example of the ‘orbit-stabiliser’ theorem. Here, under the group action of conjugation, the centraliser is the set of elements which, by commuting with element g_1 , leave g_1 unchanged or ‘stabilised’ : $g_2^{-1}g_1g_2 = g_1$. The centraliser of g_1 is therefore the special name for the stabiliser of g_1 under conjugation. Conversely, if $g_2^{-1}g_1g_2 = g_3 \neq g_1$, then g_1 and g_3 are in the same conjugacy class, from the definition of conjugacy. g_3 is in the orbit of g_1 under conjugation, meaning that g_3 can be ‘reached’ from g_1 by some path involving only conjugation.

The ‘orbit-stabiliser’ theorem is a consequence of Lagrange’s theorem because there is a bijection between the cosets of the stabiliser of an element g and the elements in the orbit of g under conjugation. As an example of how this theorem applies to D_4 , consider the element number 6 = **ab**. Its centraliser is the set of elements numbered {1, 3, 6, 8}. This means that 6 does not commute with {2, 4, 5, 7}. Observe that {2, 4, 5, 7} is the coset of {1, 3, 6, 8} and these two sets, with representative elements 2 and 1 respectively, partition G . Conjugation of 6 with any of {1, 3, 6, 8} produces only 6 (by definition of the centraliser), whilst conjugation of 6 with the coset produces only element 8 = **a³b**. Thus {6, 8} forms a conjugacy class. Element 6 in this class can be paired with the centraliser set {1, 3, 6, 8}, and element 8 paired with its coset {2, 4, 5, 7}.

Looking at the subgroups found, the program prints

```

Consolidated list of 4 proper subgroups derived from centralisers of all
elements:

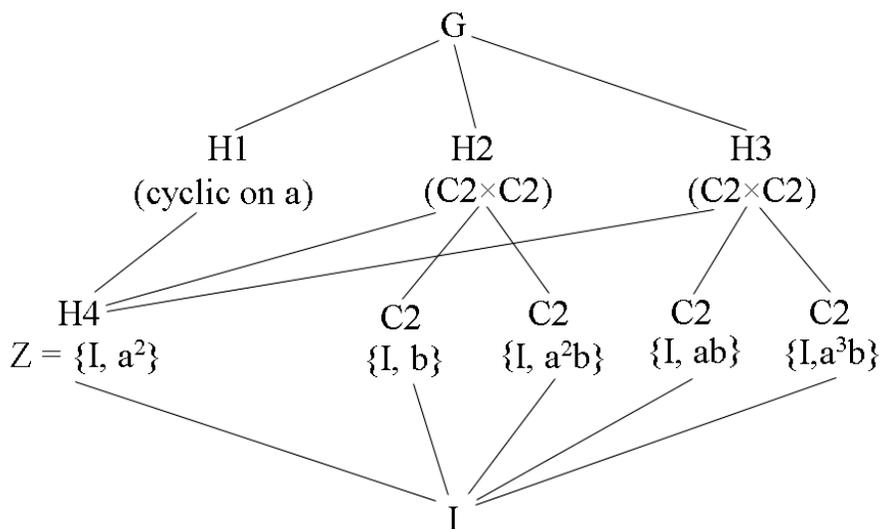
```

```

Subgroup No. 1 of 4 elements : 1  2  3  4  (H1)
Subgroup No. 2 of 4 elements : 1  3  5  7  (H2)
Subgroup No. 3 of 4 elements : 1  3  6  8  (H3)
Subgroup No. 4 of 2 elements : 1  3      (H4)

```

From this table we can construct a lattice diagram to show how one subgroup fits inside another:



This is consistent with our expected symmetries of a square and in addition shows that two reflections combine to give a direct product group $C2 \times C2$, called the *Viergruppe* (German for 4-group). You can evaluate the cosets to prove that all the order 4 subgroups are normal, but the order 2 ones are not.

The program goes on to compute the derived subgroup as elements $\{1, 3\}$, which here coincides with the centre Z . The four cosets of this form the abelianisation factor group, of type $C2 \times C2$.

Note on automorphisms. This is probably a suitable place to note that conjugation is an example of group automorphism. An automorphism of a group G is an isomorphism of G onto itself. That is, it is a bijective mapping of elements which respects the group 'product' operation, so if A is an automorphism of G , and f, g are two elements in G ,

$$A(fg) = A(f) \cdot A(g) \quad \text{where } \cdot \text{ is the group product operator.}$$

Not all bijections of the set of elements onto themselves are isomorphisms. Consider mapping an element to its inverse, $g \rightarrow g^{-1}$. This is always a bijection because every element of G has a unique inverse, but does it respect the group operation? For any elements f and g

$$(fg)^{-1} = g^{-1} \cdot f^{-1}$$

and this equals $f^{-1} \cdot g^{-1}$ only if f^{-1} and g^{-1} commute; that is, only if the group is abelian. So in non-abelian groups mapping to the inverse is not an automorphism.

Conjugation, however, is an automorphism in *every* group. To see this, note that if $A_a(x)$ means 'form the conjugate of element x with element a ' :

$$A_a(fg) = a^{-1}(fg)a = a^{-1}(f a a^{-1} g) a = (a^{-1} f a) \cdot (a^{-1} g a) = A_a(f) \cdot A_a(g).$$

The automorphisms of a group G themselves act like elements of a group (called the 'automorphism group of G ', denoted $\text{Aut}(G)$). The automorphisms effected by conjugation form a subgroup of this called the 'inner automorphism group of G ', $\text{Inn}(G)$. With this definition we can think of a normal subgroup of G as one which is invariant under all inner automorphisms. This way of looking at things was the great insight of Galois in elucidating the symmetry of the roots of polynomial equations, and hence which polynomial equations can be solved by a formula involving addition, multiplication and square roots.

So what is the inner automorphism group of D4? Use WordGroups (or PermGroups) to calculate all conjugates. For example, consider conjugation with element 2 = **a**. Under the operation $A_2(g) = (a^3) g$ a elements (by number) map as follows

$$1 \rightarrow 1, 2 \rightarrow 2, 3 \rightarrow 3, 4 \rightarrow 4, 5 \rightarrow 7, 6 \rightarrow 8, 7 \rightarrow 5, 8 \rightarrow 6.$$

This is the permutation (57)(68) in cycle notation. Similarly

$$\begin{aligned} A_1 &= A_3 = (1) = \text{Identity} \\ A_2 &= A_4 = (57)(68) \\ A_5 &= A_7 = (24)(68) \\ A_6 &= A_8 = (24)(57) . \end{aligned}$$

Thus there are only four independent automorphic permutations *i.e.* bijections from D4 to D4. The above set contains the identity plus three permutations of order 2. Moreover $(24)(68) \cdot (24)(57) = (57)(68)$. The inner automorphisms clearly form the Viergruppe, $C_2 \times C_2$. Note also that the elements 1 and 3 which map to the identity of Inn(D4) together constitute the centre, Z, of D4. In this case they also equal the commutators in the derived subgroup. The four cosets of the derived subgroup constitute the factor group D_4 / Z . So we can see that G / Z is isomorphic to Inn(G). This is a general property of inner automorphisms in any group.

Can we find the other ('external') automorphisms of D4? There are restrictions on the possibilities. For instance, any automorphism must send an element of order k to another element of order k (see footnote)³. Moreover, there is a theorem that states that Inn(G) is a normal subgroup of Aut(G). Accepting this, the order of Aut(D4) must be a multiple of 4. We get one clue how to generate Aut(D4) from observing that the permutation (24) replaces **a** by $a^3 = a^{-1}$ and a^3 by **a**, where $a = (a^3)^3 \pmod{a^4}$. Similarly permutation (68) replaces **ab** by $a^3 b$ and *vice versa*. So we might suspect that one type of automorphism replaces all **a** by a^3 , leaving the **b** unchanged. Permutation (57) replaces **b** by $a^2 b$ and $a^2 b$ by **b**. This operation would replace **ab** by $a^3 b$ and $a^3 b$ by $a^5 b = ab$, which is the permutation (68) again.. If you test whether the two operations

$$\begin{aligned} p(x) &= x \quad \text{if } x = a \\ &= a^2 x \quad \text{if } x = b \\ Q(x) &= x \quad \text{if } x = b \\ &= x^3 \quad \text{if } x = a \end{aligned}$$

together generate a group of automorphisms, you will find that p and Q both have order 2 and together generate a group of only 4 elements. This will not do because we need one generator to have order 4. Fortunately this can be arranged by defining operation P, similar to p, as follows:

$$\begin{aligned} P(x) &= x \quad \text{if } x = a \\ &= a x \quad \text{if } x = b \\ Q(x) &= x \quad \text{if } x = b \\ &= x^3 \quad \text{if } x = a \end{aligned}$$

The table below shows the effect of powers of P and Q operating on the elements of D4:

$I = P^4(x)$	$P(x)$	$P^2(x)$	$P^3(x)$	$Q(x)$	$Q(P(x))$	$P^{-1}(Q(x))$
a	a	a	a	a^3	a^3	a^3
a^2	a^2	a^2	a^2	a^2	a^2	a^2
a^3	a^3	a^3	a^3	a	a	a
b	ab	$a^2 b$	$a^3 b$	b	$a^3 b$	$a^3 b$
ab	$a^2 b$	$a^3 b$	b	$a^3 b$	$a^2 b$	$a^3 b$
$a^2 b$	$a^3 b$	b	ab	$a^2 b$	ab	$a^3 b$
$a^3 b$	b	ab	$a^2 b$	ab	b	b

³ Sketch proof. First note that an automorphism A must send the identity to the identity. If e' is the image of the identity I, and $g' = A(g)$ is the image of general element g, then isomorphism requires that $A(g) = A(I g) = A(I) A(g) = e' g' = g'$ by definition of g' , so e' is the identity in the image, and is the only element mapped from I. Second note that if g has order n it means that $g^n = I$ and $g^k \neq I$ for $k < n$. $A(g^2) = A(g \cdot g) = A(g) \cdot A(g) = [A(g)]^2$ so, by induction on the index, products map to products. Hence the only power of g which can map to the identity is g^n .

You see that P has order 4, Q order 2 and that $Q(P(x)) = P^3(Q(x))$. The eight operations $\{I, P, P^2, P^3, Q, PQ, P^2Q, P^3Q\}$ form a group of type D4. This is, in fact, the automorphism group of D4, so D4 is isomorphic to $\text{Aut}(D4)$.

To complete the analysis we should prove that i) P, Q and all their combined operations are indeed automorphisms and ii) there are no more automorphisms. Regarding i) you can see from the above table that each operation is a bijection. To see that they respect the D4 group product, consider the various cases, each of which is an example of how a product maps to a product:

$$\begin{aligned} P(a^n \cdot a^m) &= a^n \cdot a^m = P(a^n) \cdot P(a^m) \\ P(a^n b \cdot a^m) &= a^{n+1} b \cdot a^m = P(a^n b) \cdot P(a^m) \\ P(a^n b \cdot a^m b) &= a^{n+1} b \cdot a^{m+1} b = P(a^n b) \cdot P(a^m b) \\ Q(a^n \cdot b) &= a^{3n} \cdot b = Q(a^n) \cdot Q(b) \\ Q(a^n \cdot a^m) &= a^{3n} \cdot a^{3m} = Q(a^n) \cdot Q(a^m) \\ &\text{etc...} \end{aligned}$$

where \cdot denotes concatenation.

So we have found the automorphisms of D4. Note that in terms of the conjugates discussed above, $A_1 = A_3 = \text{Identity} = P^4 = Q^2$, $A_2 = A_4 = (57)(68) = P^2$, $A_5 = A_7 = (24)(68) = Q$, $A_6 = A_8 = (24)(57) = P^2Q$. Finally, if the automorphisms of a group G consist entirely of its conjugate operations in $\text{Inn}(G)$, the group is called 'complete'. D4 is not complete because it has 4 distinct inner automorphisms and 4 external ones.

Advanced Option: When the program prints the four left cosets of the commutator (derived) subgroup, you will see that the first elements are $\{I, a, b \text{ and } ab\}$. This is not a cyclic group so enter 'n'. The program asks whether it is the V-4 VierGruppe $C2 \times C2$, which it is, so type 'y'. There are four linear character which are lifted from $C2 \times C2$, leaving only the last character incomplete. (The program has built into it the character tables of $C2 \times C2$ and $C4 \times C2$ which are quite easy to recognise by eye. It does not have the facility to enter the character tables of other derived subgroups. In these cases you will have to work out the lifted linear characters yourself.)

We have some choice over how to determine the 2-dimensional character. Later we will find it by induction on a subgroup, but first let's calculate it from the linear characters using the orthogonality relations. Essentially these involve the 'dot product' of rows and columns. For complex numbers the dot product is with the complex conjugate: that is $(a \ b \ c) \times (d \ e \ f) = ad^* + be^* + cf^*$. The orthogonality relations say that :

- 1) The first column dotted with itself sums to $|G|$. For D4, $1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 2 \cdot 2 = 8$.
- 2) Each other column dotted with itself sums to $|G| / |K|$ where $|K|$ is the number of elements in the conjugacy class for that column. (This quotient equals the order of the corresponding centraliser.) Thus in column 3 the dot product equals $8/2 = 4$.
- 3) The (complex conjugate) dot product of two different columns is zero..
- 4) For a row dotted with itself, weight each term by the number of elements in that conjugacy class. Then the sum is $|G|$. Thus is row two, $1 \cdot (1^2) + 1 \cdot (1^2) + 2 \cdot (-1)^2 + 2 \cdot (-1)^2 + 2 \cdot (1)^2 = 8$.
- 5) The dot product of two different rows, weighted by the numbers in each conjugacy class, is zero.

Rule 2) applied to column 2 means that the dot product with itself equals $8/1 = 8$. The bottom value in this row must therefore be +2 or -2. Similarly in column 3 the dot product equals $8/2 = 4$. We already have $1^2 + (-1)^2 + 1^2 + (-1)^2 = 4$ so the bottom value must be 0. Similarly with the other columns. Apply rule 3) to columns 1 and 2 to see that the bottom value in column 2 must be -2. We therefore have determined the last character to be $(2 \ -2 \ 0 \ 0 \ 0)$. Rule 4) shows that this is

an irreducible character because $1*(2*2) + 1*(-2)*(-2) + 0 + 0 + 0 = 8 = |G|$. With a reducible character the weighted dot product would exceed $|G|$.

We have a further useful rule:

- 6) When all the irreducible characters are weighted by their dimension (or degree – that is, each character is multiplied by the value in its first column) and added column by column, the sum is the regular character. The regular character is $(|G| \ 0 \ 0 \ 0 \ \dots \ 0)$, which in this case is $(8 \ 0 \ 0 \ 0 \ 0)$. The regular character corresponds to representing the group elements by $|G|$ -by- $|G|$ permutation matrices (entries 0 or 1 only) and so corresponds to representing the group as a subgroup of the symmetric group on $|G|$ elements.

In this case the weighting on the linear characters is simply 1 and their sum is $(4 \ 4 \ 0 \ 0 \ 0)$. Subtract this from the regular character to get $(4 \ -4 \ 0 \ 0 \ 0)$. This must equal twice the bottom, 2-dimensional character.

Now obtain the same character by induction on two subgroups in turn. Enter ‘n’ to inputting another subgroup and ‘y’ to examining a subgroup. First select subgroup 1. The program detects that this is cyclic. Let it generate the character table from element number 2= a, and choose character (row) 2. The induced character is reported as

$$2 \quad -2 \quad 0 \quad 0 \quad 0 \quad .$$

Finally induce a character from subgroup 3, with elements $\{1, 3, 6, 8\} = \{I, a^2, ab, a^2 b\}$. The program reports that this subgroup seems not to be cyclic. Since there are only four elements, it must be the VierGruppe $C2 \times C2$. Indeed it is generated by elements a^2 and ab . The program does not determine the nature of subgroups except cyclic subgroups, but does offer the option to input a character of the subgroup if you know what it is from other considerations. In this case if a^2 has a 1-D matrix representation $[1]$ and ab is $[-1]$, then $a^3 b$ is $[-1]$ and $I = (a^2)^2$ is $[1]$. The character is the trace of these matrices.

When asked to input a character from the keyboard, enter the character value of each subgroup element as prompted. Here enter 1, 1, -1, -1 respectively. The program returns the induced character $(2 \ 2 \ 0 \ -2 \ 0)$. This is clearly not the 2-dimensional character we found above. The fact that its weighted dot product is $2^2+2^2+0+2*(-2)^2+0 = 16 = 2|G|$ confirms that this is not an irreducible character. In fact it is the sum of linear characters 2 and 3. Induction on this subgroup therefore tells us nothing new about the character table of $D4$.

Finally, if you induce from subgroup No 4 = (I, a^2) , which is the centre Z , you will get the character $(4 \ -4 \ 0 \ 0 \ 0)$. This is precisely twice the 2-D irreducible character. All characters are either irreducible or the sum of irreducibles.

2.7 Comparison of $D4$ (dihedral) with Q (quaternion)

In this exercise you are invited to compare the groups $D4$ and Q which each have eight elements and share other similarities.

To create Q enter order of a is 4, order of b is 4, two relations, first LHS = ba , first RHS = $aaab$, second LHS = bb , second RHS = aa . This sets $b^2 = a^2$. (In a strict mathematical sense the first relation is not necessary since $b^2 = a^2$ and $a^4 = I$ together imply that $b^4 = I$, but in the context of this program it is necessary.)

The list of elements for Q is the same as for $D4$ but their inverses are different, as this table shows:

Group D4			Group Q		
No.	Element	Inverse	No.	Element	Inverse
1	I	I	1	I	I
2	a	a ³	2	a	a ³
3	a ²	a ²	3	a ²	a ²
4	a ³	a	4	a ³	a
5	b	b	5	b	a ² b
6	ab	ab	6	ab	a ³ b
7	a ² b	a ² b	7	a ² b	b
8	a ³ b	a ³ b	8	a ³ b	ab

This makes the order of the elements different. D4 has

2 elements of order 4 : 2 4
 5 elements of order 2 : 3 5 6 7 8

while Q has

6 elements of order 4 : 2 4 5 6 7 8
 1 elements of order 2 : 3

This difference of orders produces a difference in the structure of the subgroups. Whereas D4 has two subgroups of order 4 with type C2×C2, in Q all subgroups are cyclic.

Advanced Option : You can show that D4 and Q have the same character tables. This fact means that the character table is not a unique identifier of a group. The difference between the groups is illustrated by the two-dimensional matrix representations of the groups. For D4 induction on subgroup number 1 gives the generating matrices

$$a = \begin{bmatrix} i & 0 \\ & \\ 0 & -i \end{bmatrix} \qquad b = \begin{bmatrix} 0 & 1 \\ & \\ 1 & 0 \end{bmatrix}$$

For Q the corresponding matrices are

$$a = \begin{bmatrix} i & 0 \\ & \\ 0 & -i \end{bmatrix} \qquad b := \begin{bmatrix} 0 & -1 \\ & \\ 1 & 0 \end{bmatrix}$$

These differs only by one minus sign in **b**, but this imparts the quaternion multiplication properties to some of the group elements. Recall that the quaternions are a four-dimensional type of complex number of the form $q = A + Bi + Cj + Dk$ where $i^2 = j^2 = k^2 = -1$ and $ij = -k, jk = -i, ki = -j$. Any of the non-commuting pairs of order-4 elements will generate quaternion properties. Here are two examples with the above matrices

Example 1 : $a^2 = b^2 = (a^3b)^2 = -1, a*b = -a^3b, b*a^3b = -a, a^3b*a = -b.$

Example 2: $(a^3)^2 = (ab)^2 = (a^2b)^2 = -1, a^3*ab = -a^2b, ab*a^2b = -a^3, a^2b*a^3 = -ab.$

2.8 An abelian semi-direct product C3 ^ C4

This is also called the dicyclic group Q6. A semi-direct product G is the product of two of its subgroups H and N where

- i) only one subgroup N is normal in G and
- ii) N and H have only the identity I in common.

We have already noted that S3 = D3 = C3 ^ C2, where C3 is normal but C2 is not.

Enter order of **a** is 4, order of **b** is 3, 1 relation, LHS is **ba**, RHS is **abb**.

There are 12 elements falling into 6 conjugacy classes. When the program gets to the abelianisation factor group, type 'y' to confirm that it is cyclic, then 'a' to state that the group is generated by **a**.

Use the program to look at the cosets of the subgroups. You will find that none with four elements, namely Nos. 2, 3 and 4, is normal. The only normal subgroup is No 6 = {I, b, b²}, which is C3, one of the product groups. This illustrates the meaning of 'semi-direct product'.

(You can also generate the group from: order of **a** is 3, order of **b** is 4, **ba**= **abb** : See Appendix).

Advanced Option : The program prints the four linear characters and shows two 2-dimensional characters. You can try to determine these from the orthogonality relations by solving the consequent simultaneous equations. It is clear from column orthogonality that the entries in the second column of both two dimensional characters must be +2 or -2, that in columns 3 and 6 the entries must to +1 or -1, and in columns 4 and 5 the entries must be zero. We therefore have characters of form

$$2 \quad \pm 2 \quad \pm 1 \quad 0 \quad 0 \quad \pm 1$$

The regular character is

$$12 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$$

and the four linear characters add to

$$4 \quad 0 \quad 4 \quad 0 \quad 0 \quad 0 \quad ,$$

so twice the sum of the 2-dimensional characters must be

$$8 \quad 0 \quad -4 \quad 0 \quad 0 \quad 0$$

or their sum is

$$4 \quad 0 \quad -2 \quad 0 \quad 0 \quad 0 \quad .$$

The only two combinations which fit are

$$\begin{array}{cccccc} 2 & -2 & -1 & 0 & 0 & \pm 1 \\ 2 & 2 & -1 & 0 & 0 & \mp 1. \end{array}$$

The choice of sign in column 6 is decided by taking the dot product of columns 2 and 6, which should be zero. The completed character table is

Group C3 \wedge C4 = Q6

Class label	K_1	K_3	K_5	K_2	K_4	K_6
Element id no.	1	3	5	2	4	9
Element	I	a ²	b	a	a ³	a ² b
Number in class	1	1	2	3	3	2
Order of element	1	2	3	4	4	6
Order centraliser	12	12	6	4	4	6

Trivial char.	1	1	1	1	1	1
	1	-1	1	i	-i	-1
	1	1	1	-1	-1	1
	1	-1	1	-i	i	-1
	2	-2	-1	0	0	1
	2	2	-1	0	0	-1

You can confirm these characters by using the program to induce characters on subgroup number 1, which has 6 elements and so will give a representation of dimension $12/6 = 2$. Select this subgroup. The program says that it is cyclic and that elements 9 and 10 will generate it. Select element 9 as the generator. The program prints

Character table of cyclic subgroup generated by element No. 9 = $a^2 b$ of order 6 is :

I	1	2	3	4	5
1	9	6	3	5	10
1	1	1	1	1	1
1	c	c ²	-1	c ^{2*}	c*
1	c ²	c ^{2*}	1	c ²	c ^{2*}
1	-1	1	-1	1	-1
1	c ^{2*}	c ²	1	c ^{2*}	c ²
1	c*	c ^{2*}	-1	c ²	c

Here $c = \exp(2\pi i/6)$, and * denotes complex conjugate.

Select row 2 of the above table as the basis for induction. The induced character is printed as

$$2 \quad -2 \quad c^2+c^{2*} \quad 0 \quad 0 \quad c+c^*$$

which becomes

$$2 \quad -2 \quad -1 \quad 0 \quad 0 \quad 1$$

when the value of c is substituted.

Now select to induce another character from the same subgroup and this time select row 3. The induced character is

$$2 \quad 2 \quad c^{2*}+c^2 \quad 0 \quad 0 \quad c^2+c^{2*}$$

or

$$2 \quad 2 \quad -1 \quad 0 \quad 0 \quad -1$$

which is the other irreducible non-linear character. Check that it is irreducible by evaluating its weighed dot product with itself : $1*2^2 + 1*2^2 + 2*(-1)^2 + 0 + 0 + 2*(-1)^2 = 12 = |G|$.

If you induce a character from the cyclic subgroup number 2 with 4 elements, you obtain the character

$$3 \quad -3 \quad 0 \quad i \quad -i \quad 0.$$

This must be reducible and in fact is the sum of rows 2 and 5 in the character table:

$$\begin{matrix} 1 & -1 & 1 & i & -i & -1 \\ 2 & -2 & -1 & 0 & 0 & 1 \end{matrix}$$

Finally, if you use the transformation matrix T given in §2.4 on the induced 3-dimensional matrices for **a** and **b**, you will find that they, and all other group elements, are converted to block diagonal form. For example,

$$a = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} \quad \text{and} \quad T^{-1}aT = \begin{bmatrix} i & 0 & 0 \\ 0 & i & -i \\ 0 & 0 & -i \end{bmatrix}$$

They give the characters $(1 \quad -1 \quad 1 \quad i \quad -i \quad -1) + (2 \quad -2 \quad -1 \quad 0 \quad 0 \quad 1)$, consistent with our deduction above.

2.9 An Alternating Group, A4

This and the next exercise concern the symmetric group S_4 (24 elements) and one of its subgroups, the alternating group, A_4 (12 elements). Physically, S_4 is represented by all the $4!$ permutations of 4 elements. The group A_4 is made up of all permutations which can be composed from an even number of transpositions. When taken together with its one coset, which is the odd-order permutations, A_4 is augmented to form S_4 .

A_4 is the symmetry group of rotations of the regular tetrahedron, so A_4 is sometimes called the chiral tetrahedral group. (If mirror reflections are added, the 4 vertices of the tetrahedron can be arranged in every permutation and its symmetry group become S_4 .) To picture the subgroups of symmetries consider the following

- The tetrahedron has four equal faces each of which is an equilateral triangle, and hence the solid has four equivalent 3-fold rotation axes. Each such axis is at right angles to a face and passes through the fourth vertex. A 120° rotation about, say vertex 1, will cycle vertices 2, 3 and 4, leaving vertex 1 fixed. In cycle notation this corresponds to $(1)(234)$, which involves precisely two transpositions. Since rotation can be either clockwise or anticlockwise, we expect twice $4 = 8$ elements of order 3 in the group.
- The three 2-fold rotation axes are found as follows. Take a pair of vertices, 1 and 2 say, and find the midpoint of the edge which connects them. Do the same with the other two vertices, 3 and 4. The edges 1-2 and 3-4 are perpendicular and the line which joins their midpoints is perpendicular to both of them. This third line is a 2-fold axis rotation axis.

It is possible to present A_4 as a group of words on two generators \mathbf{a} and \mathbf{b} subject to $\mathbf{a}^2 = \mathbf{I}$, $\mathbf{b}^3 = \mathbf{I}$, $(\mathbf{ba})^3 = \mathbf{I}$. However the relation $(\mathbf{ba})^3 = \mathbf{I}$ cannot be entered as such into WordGroups. Recall that the program generates all strings of \mathbf{a} and \mathbf{b} , so will produce $\mathbf{bbabbabba}\dots$. It is therefore necessary to derive from $(\mathbf{ba})^3 = \mathbf{I}$ two relations which ensure words of finite length. Moreover, in each relation the LHS must start in \mathbf{b} and end in \mathbf{a} , and the RHS must start in \mathbf{a} and end in \mathbf{b} . Hence in the program enter:

```

order of a is 2,
order of b is 3,
2 relations,
LHS = baba, RHS = abb,
LHS = bba, RHS = abab.

```

The correct 12 elements will now be generated.

The alternating and symmetric groups are cases in which it is easier to see the group as built from permutation cycles using PermGroups. A_4 is the group of all even transpositions on 4 elements. There are two types of even transpositions: ones which are disjoint pairs of 2-cycles, and ones which are overlapping pairs of 2-cycles. Hence take $\mathbf{a} = (12)(34)$ (disjoint, order 2) and $\mathbf{b} = (13)(12) = (123)$ (overlapping, order 3). Products of these are the following group elements :

No.	Word	Permutation	Inverse
1	\mathbf{I}	$(1)(2)(3)(4)$	\mathbf{I}
2	\mathbf{a}	$(12)(34)$	\mathbf{a}
3	\mathbf{b}	(123)	$(132) = \mathbf{b}^2$
4	\mathbf{b}^2	(132)	$(123) = \mathbf{b}$
5	\mathbf{ab}	(243)	$(243) = \mathbf{abab}$
6	\mathbf{ab}^2	(143)	$(134) = \mathbf{ba}$
7	\mathbf{ba}	(134)	$(143) = \mathbf{ab}^2$
8	\mathbf{abab}	(234)	$(243) = \mathbf{ab}$

9	aba	(142)	(124) = bab
10	bab	(124)	(142) = aba
11	bab ²	(14)(23)	baba ²
12	abab ²	(13)(24)	abab ²

You can see various conjugacy classes from the different cycle types.

Let's look at another example of the orbit-stabiliser theorem here. With conjugation as the operation this is the 'conjugacy class-centraliser theorem'. It says that the order of the centraliser multiplied by the number of elements in the conjugacy class equals the group order $|G|$. Take, for example, element $5 = ab$. Its centraliser has elements $\{1, 5, 8\}$ and the cosets of this subgroup are $\{2, 4, 9\}$, $\{3, 6, 12\}$ and $\{7, 10, 11\}$. Under conjugation all the elements in $\{2, 4, 9\}$ send 5 to element 7, all those in $\{3, 6, 12\}$ send 5 to 9, and all those in $\{7, 10, 11\}$ send 5 to 3. The conjugacy class containing 5 is $\{3, 5, 7, 9\}$, and each coset of the centraliser sends 5 to a different member of this class. This is a bijection.

The derived subgroup is calculated to be $\{I, a, bab^2, abab^2\}$. By looking at the cosets of the various subgroups, checks that this is the only normal subgroup. Enter an additional subgroup $\{I, a\}$ from the keyboard and show that this too is not normal.

Of the five groups of order 12, A_4 is the only one without a subgroup of degree 6. In particular S_3 is not a subgroup of A_4 because S_3 has elements such as (12) which have an odd number of transpositions.

Advanced Option : The three cosets of the derived subgroup have representative elements (transversal) $\{I, b, b^2\}$ so type 'y' to their forming a cyclic subgroup, and 'b' as the letter involved. The program calculates the character table of this abelianisation factor group, C_3 , and lifts its characters into the character table for A_4 . The table at this stage shows a single incomplete character of degree 3. This occurs because $1^2+1^2+1^2+3^2$ is the only combination of square integers to equal $12 = |G|$.

This character, which must be faithful, can be found from the orthogonality relations. Suppose it is $(3 \ p \ q \ r)$ where p, q and r are to be found. The (complex) dot product of the second column with itself is 4, so $p = \pm 1$. The dot product of the first and second columns, being zero, shows that $p = -1$. The dot product of the third column with itself is 3 (order of centraliser) so q must be 0 since $|e| = 1$. Similarly with r . The completed character table of A_4 is therefore

Group A4				
Class label	K1	K2	K3	K4
Element id no.	1	2	3	4
Element	I	a	b	b ²
Number in class	1	3	4	4
Order of element	1	2	3	3
Order centraliser	12	4	3	3
<hr/>				
Trivial char.	1	1	1	1
	1	1	e	e*
	1	1	e*	e
	3	-1	0	0
<hr/>				

where $e = \exp(2\pi i/3)$.

We will now obtain the degree-3 character by induction on a subgroup. The 7th subgroup listed is the derived or commutator subgroup, which happens to be identical to subgroup 1. Select either copy of this subgroup. It has elements {1, 2 11, 12}. The program says that it is not cyclic, so it must be C2×C2. Say ‘y’ to calculate multiples of elements and type 2, <RTN>, 11,<RTN>, 1 <RTN> to calculate the product of elements numbers 2, 11, 1; this is 2×11 = element 12 since element 1 = I. Now elect to enter a character from the keyboard and type 1, 1, -1, -1 to each respective prompt. This enters the character value 1 for element 1, 1 for element 2, -1 for element 11 and -1 for element 12. The induced character is (3 -1 0 0) as required.

It is worth looking at the 3-D matrices which represent **a** and **b**:

$$\mathbf{a} = \begin{bmatrix} 1 & 0 & 0 \\ & & \\ 0 & -1 & 0 \\ & & \\ 0 & 0 & -1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 & 0 & 1 \\ & & \\ 1 & 0 & 0 \\ & & \\ 0 & 1 & 0 \end{bmatrix}$$

Consider these operating by matrix multiplication on three mutually perpendicular vectors x, y and z . **a** would leave x unmoved, but reverse the directions of y and z . This is equivalent to a 180° rotation about the axis x . Elements 11 and 12 behave similarly, keeping axes y and z respectively fixed. Matrix **b** causes x, y and z to cycle into the order z, x, y . All the rotational symmetry properties of the tetrahedron are represented by the twelve matrices built from **a** and **b**.

In the program if you select subgroup 4, you will see that it is cyclic, C3, generated by either element 6 or 7. If you induce a character on this subgroup, it will have degree $12/3 = 4$. If you select the character in row 2 of the C3 table, you will get the induced character of A4 to be {4 0 e* e}. This is the sum of the two irreducible characters (3 -1 0 0) + (1 1 e* e).

A4 has some distinguishing features. It is the smallest group with an irreducible 3-dimensional representation, and the largest group with only four conjugacy classes (and hence four irreducible characters). Of the five groups of order 12 it is the only one without a subgroup of degree 6.

2.10 The symmetric group, S4

Like its subgroup A4, S4 can be generated using WordGroups from **a** and **b** using the relations $\mathbf{a}^2 = \mathbf{I}$, $\mathbf{b}^4 = \mathbf{I}$, $(\mathbf{ba})^3 = \mathbf{I}$. However, as with A4, it is necessary to prevent the program generating infinite strings of **babababa....**, **bbabbabba...** and **bbbabbbbabbba....**. Hence enter:

- order of **a** is 2,
- order of **b** is 4,
- 3 relations with
 - 1 LHS = **baba**, RHS = **abbb**,
 - 2 LHS = **bbba**, RHS = **abab**,
 - 3 LHS = **babba**, RHS = **abbabb**.

The program will now generate the 24 elements of S4. I suggest you elect to have a thorough check on consistency when prompted.

It is easier to see how S4 is generated by using permutations on 4 elements. This readily done using PermGroups, but here observe that if **a** = (12) and **b** = (1234) we have:

No.	Element	Permutation	Inverse	Inverse No.
1	I	(1)(2)(3)(4)	I	1
2	a	(12)	a	2
3	b	(1234)	b ³	5
4	b ²	(13)(24)	b ²	4

5	b^3	(1432)	b	3
6	ab	(234)	$abab$	11
7	$a b^2$	(1324)	$b^2 a$	10
8	$a b^3$	(143)	ba	9
9	ba	(134)	ab^3	8
10	$b^2 a$	(1423)	ab^2	7
11	$abab$	(243)	ab	6
12	aba	(1342)	bab	14
13	$a b^2 a$	(14)(23)	$ab^2 a$	13
14	bab	(1243)	aba	12
15	bab^2	(142)	$b^2 ab^3$	19
16	bab^3	(23)	bab^3	16
17	$b^2 ab$	(132)	$aba b^3$	21
18	$b^2 ab^2$	(34)	$b^2 ab^2$	18
19	$b^2 ab^3$	(124)	bab^2	15
20	$abab^2$	(14)	$abab^2$	20
21	$abab^3$	(123)	$b^2 ab$	17
22	$ab^2 ab$	(13)	$ab^2 ab$	22
23	$ab^2 ab^2$	(12)(34)	$ab^2 ab^2$	23
24	$ab^2 ab^3$	(24)	$ab^2 ab^3$	24

There are

9 elements of order 2 :	2	4	13	16	18	20	22	23	24
6 elements of order 4 :	3	5	7	10	12	14			
8 elements of order 3 :	6	8	9	11	15	17	19	21	

You can see that elements 2, 16, 18, 20, 22, 24 have a different cycle structure from elements 4, 13 and 23, even though both sets have order 2. They must be in different conjugacy classes. Indeed the list of the five conjugacy classes confirms this:

Class 1 has 1 elements of order 1:	1								
Class 2 has 6 elements of order 2:	2	16	18	20	22	24			
Class 4 has 3 elements of order 2:	4	13	23						
Class 5 has 8 elements of order 3:	6	8	9	11	15	17	19	21	
Class 3 has 6 elements of order 4:	3	5	7	10	12	14			

We can use a little combinatorics to find something interesting about the number of conjugacy classes in any symmetric group S_N . The number of conjugacy classes in S_N equals the number of partitions of the integer N . A partition of N is a way of splitting N into the sum of lesser integers. Here is a table of all the partitions of integers 4, 5, 6 and 7:

4	5	6	7
3 1	4 1	5 1	6 1
2 2	3 2	4 2	5 2
2 1 1	3 1 1	4 1 1	5 1 1
1 1 1 1	2 2 1	3 3	4 3
	2 1 1 1	3 2 1	4 2 1
	1 1 1 1 1	3 1 1 1	4 1 1 1
		2 2 2	3 3 1
		2 2 1 1	3 2 2
		2 1 1 1 1	3 2 1 1
		1 1 1 1 1 1	3 1 1 1 1
			2 2 2 1
			2 2 1 1 1
			2 1 1 1 1 1
			1 1 1 1 1 1 1
$P(4) = 5$	$P(5) = 7$	$P(6) = 11$	$p(7) = 15$

Suppose we associate with each partition a unique cycle structure. For example $7 = 3+3+1$ corresponds with the permutation $(abc)(def)(g)$ in S_7 . Now each conjugacy class has a distinct cycle type. In S_N all possible permutations occur, so all possible cycle types will also occur. So there is a bijection between the conjugacy classes and the partition of N . We therefore expect S_4 to have five conjugacy classes, which indeed is what WordGroups and PermGroups calculate..

The centre Z of S_4 is trivial; $Z = I$. The derived subgroup has 12 elements and you should find that it has the structure of A_4 . This is a normal subgroup. The program lists 14 proper subgroups and clearly there are others such as $\{I, b^2\}$. I leave it to you to construct a complete lattice diagram showing how all the subgroups fit together inside S_4 .

As a window into more advanced group theory, let us look back over the three groups S_4 , A_4 and Viergruppe = $C_2 \times C_2$. A_4 is a normal subgroup of S_4 , and $C_2 \times C_2$ is a normal subgroup of A_4 . We come to the ‘composition series’ of nested normal subgroups:

$$I \subset C_2 \subset C_2 \times C_2 \subset A_4 \subset S_4.$$

In this series it is not possible to ‘insert’ another normal subgroup between any two neighbours, so in this sense each normal subgroup as big as possible. That makes the quotient groups of adjacent pairs as small as possible. The quotient subgroups are called the ‘composition factors’ of S_4 and are:

$$C_2 / I = C_2, \quad C_2 \times C_2 / C_2 = C_2, \quad A_4 / (C_2 \times C_2) = C_3, \quad S_4 / A_4 = C_2.$$

Multiply the orders of these quotient groups and find that $2 \times 2 \times 3 \times 2 = 24 = |S_4|$. A group G is said to be ‘soluble’ or ‘solvable’ if G has a composition series of normal subgroups for which all the quotient groups (the composition factors) are abelian. This is the case with S_4 . The ‘simple’ groups have no composition series other than I , and in this sense they are ‘prime’ or irreducible.

Advanced Option: The program produces an incomplete character table with five characters, only two of which are linear.

Class label	K1	K2	K4	K5	K3
Element id no.	1	2	4	6	3
Element	I	a	b ²	ab	b
Number in class	1	6	3	8	6
Order of element	1	2	2	3	4
Order centraliser	24	4	8	3	4
<hr/>					
Trivial char. R1	1	1	1	1	1
R2	1	-1	1	1	-1
R3	2	n	p	q	r
R4	3	s	t	u	v
R5	3	w	x	y	z
	C1	C2	C3	C4	C4

Above I have filled the blanks with letters. Let us try to find some of these missing values using the orthogonality relations. Take the weighted dot product of rows R1 and R2 (weighted by the number of elements in each class), and of R2 and R3

$$\begin{aligned} 2 + 6n + 3p + 8q + 6r &= 0 \\ 2 - 6n + 3p + 8q - 6r &= 0 \end{aligned}$$

so $2 + 3p + 8q = 0$ and $n = -r$
 Similarly from rows R4 and R5

$$2 + 3t + 8u = 0 \text{ and } s = -v$$

$$2 + 3x + 8y = 0 \text{ and } w = -z .$$

Take the (complex) dot product of column C2 with itself. It equals the order of the centraliser for that conjugacy class. Hence

$$1 + 1 + |n|^2 + |s|^2 + |w|^2 = 4.$$

The modulus signs allows for the possibility of n etc. being complex. I risk some intuition, even guesswork here. I suspect that the modulus of the missing values is an integer, and that the two 3-D characters in rows 4 and 5 share some common features, making row 3 the ‘odd one out’. Hence I conjecture that one of n, s, w is 0, that probably it is n which is 0, and that the other two have modulus 1. In a similar fashion, C3 and C4 dotted with themselves give

$$|p|^2 + |t|^2 + |x|^2 = 6.$$

$$|q|^2 + |u|^2 + |y|^2 = 1.$$

Taking R3 as the odd one out, I conjecture that $p = \pm 2, t = \pm 1, x = \pm 1, q = \pm 1, u = y = 0$.

Tentatively, the character table now looks as follows:

R1	1	1	1	1	1	
R2	1	-1	1	1	-1	
R3	2	0	± 2	± 1	0	
R4	3	s	± 1	0	-s	with s, w = ± 1
R5	3	w	± 1	0	-w	
	C1	C2	C3	C4	C4	

Now use the relation $2 + 3p + 8q = 0$. The solution in integers is $p = 2, q = -1$. Similarly $2 + 3t + 8u = 0$ means $t = -1$, and $2 + 3x + 8y = 0$ means that $y = -1$.

The dot product of columns C1 and C2 requires that $s = -w$. You can check that these conjectured relations do indeed satisfy all the orthogonality relations, and that their weighted sum equates to the regular character (24 0 0 0 0). The completed character table is

Group S4						
Class label		K1	K2	K4	K5	K3
Element id no.		1	2	4	6	3
Element		I	a	b^2	ab	b
Trivial char. R1	1	1	1	1	1	1
R2	1	-1	1	1	-1	-1
R3	2	0	2	-1	0	0
R4	3	1	-1	0	-1	-1
R5	3	-1	-1	0	1	1

In §3.5 of Part 3 there is a description of how to use the permutation character to find one of the order-3 characters very easily.

Now you come to the 'structure of subgroups and induced characters' section of the program. It lists 15 subgroups. Reply 'n' to 'do you wish to enter a new subgroup from the keyboard?', then select subgroup No. 15 with 12 elements; this is the derived subgroup. We can expect to obtain a 2-dimensional character by induction from this since $|G|/12 = 2$, though whether it will be the 2-D irreducible character remains to be seen. Subgroup No. 15 is not cyclic so we need to determine its nature, find its character table and enter one or more characters from the keyboard.

To find the orders of the elements in subgroup No. 15 you can either refer back to list at the beginning of the print out or use calculator offered at 'do you wish to calculate multiples of elements?' There are

3 elements of order 2 : (4, 13, 23)
 8 elements on order 3 : (6, 8, 9, 11, 15, 17, 19, 21).

Use the facility to calculate products of elements to find the conjugates of the order-3 elements. For example, entering 4 <RTN> 11 <RTN> 4 <RTN> shows that a conjugate of 11 is 15. From this evidence it is apparent that this is the group A4 with conjugacy classes (1), (4,13,23), (6, 8, 19, 17), (9, 11, 15, 21). Reference to the header of the A4 character table shows that these can be matched with elements I, A, B and B² respectively in that group, where I have used capitals to avoid confusion with the elements of S4.

We can now select the character in A4 to induce up to S4. Choose (1 1 e e*) where $e = \exp(2\pi i/3)$. So in subgroup 15, elements 4, 13 and 23 are given character value 1, elements 6, 8, 19, 17 get value e, and 9, 11, 15, 21 get value e*. Enter these when prompted. The program responds

```

Matrix representation of 'a' is           Matrix reprn. of 'b' is
      0   1                               0   e*
      1   0                               e   0

Induced character is
2   0   2   e+e*   0 .

```

This is the required 2-D irreducible character. But look at the 2-D matrices representing **a** and **b**. You can readily show that **b**² is the unit matrix. However the order of **b** has been entered as 4, not 2. So both **b**² and **b**⁴ share the same matrix in this 2-D representation. It is not faithful.

To find the 3-D characters of S4 we expect to have to induce from a subgroup of order $24/3 = 8$, so select subgroup number 1 with elements 1, 3, 4, 5, 13, 22, 23, 24. Again, by calculating powers and conjugates of elements you will be able to identify this subgroup with the dihedral group D4, studied in §2.5 The header of the character table of D4 shows conjugacy classes with representative elements I, A², B, AB, A, and these can be matched with the elements of subgroup 1 as follows:

I : 1
 A² : 4
 B : 13, 23
 AB : 22, 24
 A : 3, 5.

First let's choose to induce from row 2 of the table for D4. As prompted by the program, enter these values against the elements of subgroup 1: 1, 1, 1, 1, -1, -1, -1. The induced character is

$(3 \ -1 \ -1 \ 0 \ 1)$, which agrees with row 5 in the S4 table deduced above. I leave it to you to check whether this is a faithful character (you could multiply the 3-D matrices for a and b).

Next induce from row 4 of the table for D4. As prompted by the program, enter these values against the elements of subgroup 1: 1, -1, 1, -1, -1, 1, -1, 1. The induced character is $(3 \ 1 \ -1 \ 0 \ -1)$, which agrees with row 4 in our S4 table. Again I leave it to you to check whether this is faithful.

As a final investigation of S4, let us find a set of representative permutation matrices which make plain the fact that S4 is all the permutations of 4 items. Recall that a permutation matrix has entries which are only either 0 or 1, with one 1 per row and one per column. We might expect this to be 4-by-4 matrix. To find a 4-D representation we need to induce from a subgroup of order $24/4 = 6$. However the program has not listed a subgroup of order 6, so we will have to find one ourselves.

It is clear that S4 must have four copies of S3 as subgroups, corresponding to the permutation of three of the four items, with the fourth held constant. You can investigate which elements form these four S3 subgroups using the calculational facilities of the program. One has elements $\{1, 6, 11, 16, 18, 24\}$. Induce from the trivial character of S3 and obtain the matrices

$$\begin{array}{l}
 \begin{array}{c}
 a = \begin{bmatrix}
 0 & 1 & 0 & 0 \\
 [& & &] \\
 1 & 0 & 0 & 0 \\
 [& & &] \\
 0 & 0 & 1 & 0 \\
 [& & &] \\
 0 & 0 & 0 & 1
 \end{bmatrix}
 \end{array}
 \qquad
 \begin{array}{c}
 b = \begin{bmatrix}
 0 & 0 & 0 & 1 \\
 [& & &] \\
 1 & 0 & 0 & 0 \\
 [& & &] \\
 0 & 1 & 0 & 0 \\
 [& & &] \\
 0 & 0 & 1 & 0
 \end{bmatrix}
 \end{array}
 \end{array}$$

These correspond to the permutations $\mathbf{a} = (12)$ and $\mathbf{b} = (1234)$, in cycle notation. Using either multiplication of the matrices or combination of the permutation cycles, you can show that element number 6 = \mathbf{ab} is (234), No. 7 = \mathbf{ab}^2 is (1324), No. 8 = \mathbf{ab}^3 is (143) and No. 9 = \mathbf{ba} is (134), the inverse of 8, etc.

Incidentally, the induced character is $(4 \ 2 \ 0 \ 1 \ 0)$, which is the sum of rows 1 and 4 in the S4 table.

2.11 Five groups of order 20.

This final exercise with WordGroups is to examine all the five groups of order 20. The table in the Appendix indicates that, in addition to the cyclic group C20, groups with 20 elements may be generated by relations

$$\begin{array}{l}
 10 \ 2 \ ab \\
 10 \ 2 \ a^9b. \\
 5 \ 4 \ ab \\
 5 \ 4 \ a^2b. \\
 5 \ 4 \ a^3b. \\
 5 \ 4 \ a^4b.
 \end{array}$$

where the meaning is that $\mathbf{a}^{10} = \mathbf{I}$, $\mathbf{b}^2 = \mathbf{I}$, $\mathbf{ba} = \mathbf{ab}$, and similarly.

I leave it to you to explore these and their subgroups. You will find that the five groups are:

- The cyclic group C_{20} , produced by entering [20 20 b=a]
- The direct product $C_5 \times C_4$ produced from [5 4 ba=ab] and isomorphic to C_{20}
- The direct product $C_{10} \times C_2$, produced from [10 2 ba=ab]
- The dihedral D_{10} , produced from [10 2 ba=a⁹b]
- The dicyclic Q_{10} , which is the semi-direct product $C_5 \rtimes C_4$, produced by [5 4 a⁴b].
- A group called the holomorph of C_5 produced from either [5 4 ba=a²b] or [5 4 ba=a³b].
This is the semidirect product of C_5 with its automorphism group : $C_5 \rtimes \text{Aut}(C_5)$.

These five groups provide an opportunity to examine the Sylow theorems. These are a partial converse of Lagrange's theorem – that the order of each subgroup H divides the order of the parent group G . For $|G| = 20$ Lagrange tells us that the orders of all proper non-trivial subgroups must be 2, 4, 5, or 10, but there is no guarantee that a subgroup of order 2, 4, 5, or 10 *must* be present. In contrast, the Sylow theorems give partial information on which orders of subgroup must be present.

The key to Sylow's theorems is to consider the prime factors of $|G|$ and their powers which divide $|G|$. For each prime factor p there will be a maximal power p^m such that p^m divides $|G|$ but p^{m+1} does not. Any subgroup with maximal order p^m is called a Sylow p -subgroup. For $|G| = 20$ the primes are 2 and 5, and their maximal powers are 2^2 and 5^1 . The first Sylow theorem states that there exists at least one p -subgroup for every prime factor p . Therefore all five groups listed above must each have at least one subgroup of order 4 and one of order 5. Note that the theorem makes no statement about subgroups of order 2.

The third Sylow theorem takes this further by stating something about the number of Sylow p -subgroups for each p . The number, K , a) is a factor of g and b) has the form $1+kp$ where k is a positive integer. This severely limits the numbers of options for K . For $|G| = 20$ and $p = 2$, $1+kp = 1, 3, 5, 7, 9, 11, \dots$ from which only 1 and 5 divide 20. Therefore the number of subgroups of order 4 is either 1 or 5. For $p = 5$, $K = 1, 6, 11, \dots$ and only 1 divides 20 so the subgroup of order 5 is unique. Even stronger, a unique subgroup of order p must conjugate to itself so its left cosets are the same as its right cosets – that is, the subgroup is normal.

The second Sylow theorem states that where there are several p -subgroup belonging to the same prime p , they are all conjugate to each other. For any group of order 20, subgroups H with $|H| = 4$ must therefore all be either isomorphic to C_4 or to $V = C_2 \times C_2$, and never of mixed type.

That concludes the exercises and examples with WordGroups.

PART 3 : PermGroups : Examples and Exercises

Note that greater detail in running most examples is given in the parallel set of notes on WordGroups. I recommend that you work mainly from Part 2. This Part describes aspects peculiar to PermGroups.

Recall that in this program cycles are entered without brackets and disjoint cycles are separated by any of the characters

' / \ * x . ,

Thus $(123)(567)$ could be entered as `123'567` or `123\567` etc.

Note that when the program generates further elements from products of all elements found so far, it takes note of the time taken to find the most recent element and then goes on calculating for as long again. This is a compromise in calculational thoroughness since to search exhaustively for every possible combination can take a long time when the group is large. Its effect is to cause the program to appear to pause for a while before outputting a list of the elements.

3.1 A cyclic group C5

This example is given in some detail, to get you used to running PermGroups.

C5 is the cyclic group of order 5 (also called Z_5). Before running the program create a folder for your output called, say, Perm on the C: drive. Start the program and input 'y' to save the output to a file. You will have to enter the full path of the file, e.g. C:\Perm\C5. The program will save this as C:\Perm\C5_***.rtf where *** are three random characters.

Select whether to have the scroll lock on or off, then enter, as prompted, the number N of digits on which the permutations will be based. Since N digits could allow S_N with $N!$ elements to be generated, the program allocates an array of size $N!$ in readiness.. For $N = 7$ or greater you are asked to confirm that such a large array is indeed necessary and, if not, you can specify a suitably smaller array. To generate C5 enter '5'.

When prompted, enter the N symbols one at a time, with <RTN> following each. For this example enter the digits 1, 2, 3, 4, 5. However the program will accept any strings.

The program has the option to calculate products of permutation cycles working in the order right-to-left or left- to-right. You are asked to confirm with 'y' that right-to-left is acceptable; typing 'n' will cause multiplication to proceed from left to right.

You are now asked to enter how many seed permutations you will enter. For C5 we only need one, so enter 1. When prompted, enter 12345.

The program forms all products and inverses, and prints to screen the number of elements found so far. Those prefixed by a _ character have been found as the inverse of the previous character. It then prints a summary list of group elements and their inverses as shown below. The elements are numbered in the order they are found. You will refer to elements by these numbers in interacting with the program.

The summary table of all elements:reads:

No.	Element	Inverse	Inv No.
1	1	1	1
2	12345	15432	5
3	13524	14253	4
4	14253	13524	3
5	15432	12345	2

Order of group is 5. There are
 1 elements of order 1 : 1
 4 elements of order 5 : 2 3 4 5
 Group is cyclic, generated by element 2 = 12345

You are then offered the option of calculating products of elements. Try it by typing 'y'. Type '2, 2, 1' or '2 <RTN>, 2 <RTN> 1 <RTN>'. It replies that the product is 13524 which is element number 3. What this means is that element 2 (12345) concatenated with itself then with element 1 (identity) gives (14253) To close the product calculator enter '-1 <RTN><RTN> <RTN>' and move to the next stage.

The program now finds all the elements which commute with a chosen element. Recall that f and g commute when $fg = gf$. It takes each element g in turn and determines its order, then tests g with all elements in the group in turn to see if they commute with g . The set of all elements which commute with g is called the 'centraliser' of g .

You will see this same set of five centraliser elements appearing against every group element. This is the nature of a cyclic group. At the end of the list of centralisers the program identifies the centre Z as the whole group. The centre Z of a group is the set (which is also a normal subgroup) which commutes with every element in G . For a cyclic group Z is the whole group.

The next stage of the PermGroups works out the conjugacy classes. It takes each element g_1 in turn and combines it with every other element g_2 , recording for each combination the resulting element. Elements which are conjugate to one another form a conjugacy class. The program collates these classes, first putting them in order of the element which created them, and then in order of increasing order of element. (Sorry about the two meanings of 'order' there!) Because in a cyclic group each element commutes with every other, each element is conjugate only to itself. There are therefore five conjugacy classes.

The program now calculates the commutator elements of G . These are elements which can be 'factored' into the form $g^{-1}h^{-1}gh$. For group C5 there is only one such element – the identity I . In some groups the set of commutators is not necessarily closed so the program calculates powers and multiples of the elements to extend the set to a full subgroup. It reports whether elements have been added. This subgroup made from commutators is the 'commutator' or 'derived' subgroup. For C5 it is the trivial group $\{I\}$. The program calculates the cosets of the derived subgroup.

The program now comes to a section headed 'Structure of subgroups and induced characters'. It lists the subgroups found so far from the centralisers of the elements and the asks if you wish to enter another subgroup from the keyboard. Since the only subgroups of C5 are $\{I\}$ and C5 itself, there is little point, so type 'n'. It then asks if you wish to examine a subgroup. Enter 'n' again. Decline all further options. That concludes our basic-level investigation of C5.

Advanced option: When you are asked whether you wish to include the more advanced material on group representations, reply 'y'.

The goal of the analysis is to produce the group character table and identify most of its subgroups.

When the cosets of the derived subgroup are printed, you are asked ‘do the first elements of the cosets form a cyclic subgroup of order 5?’. They clearly do, so type ‘y’. It asks for the number of an element which will generate this cyclic group, so type ‘2’. The program calculates the character table of the cyclic group. It uses the notation e to stand for $\exp(2\pi i/5)$, e^* for $\exp(-2\pi i/5)$, e^2 for $e^2 = e^2$, etc.

In the case of C_5 all the five characters are linear (one-dimensional). Here is the complete character table for C_5 . The five conjugacy classes are labelled K1 to K5. Note that the conjugacy classes are not in the same order as when this table is generated by the companion program WordGroups.

Group C5 (Z5)					
Class label	K1	K2	K3	K4	K5
Element id no.	1	2	3	4	5
Element	1	12345	13524	14253	15432
Number in class	1	1	1	1	1
Order of element	1	5	5	5	5
Order centraliser	5	5	5	5	5
<hr/>					
Trivial char.	1	1	1	1	1
	1	e	e ²	e ^{2*}	e*
	1	e ²	e*	e	e ^{2*}
	1	e ^{2*}	e	e*	e ²
	1	e*	e ^{2*}	e ²	e

3.2 Abelian direct products, $C_m \times C_n$

A ‘direct product’ group is one in which the elements are the Cartesian products of the elements in each constituent group. The permutation approach is particularly suitable to direct products of cyclic groups because these are generated by disjoint cycles. For instance for $N = 4$, the seed permutations (12) and (34) will generate $C_2 \times C_2$, the *Viergruppe*. Whereas WordGroups, having only two generators, is limited to products of two cyclic groups, there is in principle no limitation with PermGroups.

Where the component groups are cyclic, C_m, C_n , etc., the direct product is abelian. The order of the group is merely the product of the orders of its subgroup factors: $|G| = mn\dots$. The constituent groups all become normal subgroups of order m, n , etc.. The group character table (in *Advanced Option*) has $|G|$ conjugacy classes and $|G|$ characters (row). Both WordGroups and PermGroups will give explicit character tables of $C_m \times C_n$ composed of two disjoint cycles. Character tables for three or more cycles can be large, though their structure is essentially the simple nesting of one character table within another.

3.3 A symmetric group S_3 = a dihedral group D_3

The symmetric group S_N is comprised of all permutations of N items and has $N!$ elements. S_3 is identical with D_3 and has $3! = 6$ elements. It is the smallest non-abelian group.

Generate the group by entering three symbols 1, 2, 3. Enter two seed permutations, (123) and (12). The program generates a group of 6 elements which divide into three conjugacy classes:

List of the 3 conjugacy classes sorted in ascending order of element:

```

Class 1 has 1 elements of order 1:    1
Class 3 has 3 elements of order 2:    3  5  6
Class 2 has 2 elements of order 3:    2  4

```

The three commutators is the commutator (derived) subgroup are

```

  1    2    4
  1  123  132

```

The program lists the subgroups found from the centralisers of the elements. These are printed as

```

Subgroup No. 1 of 3 elements : 1  2  4 (= derived subgroup)
Subgroup No. 2 of 2 elements : 1  3
Subgroup No. 3 of 2 elements : 1  5
Subgroup No. 4 of 2 elements : 1  6
Subgroup No. 5 of 1 elements : 1

```

If you select the option of evaluating the left and right cosets of the derived subgroup, you will find that, as sets, they are identical:

```

Left cosets of subgroup No. 1 are :
{ 1  2  4 }
{ 3  6  5 }
Right cosets of subgroup No. 1 are :
{ 1  2  4 }
{ 3  5  6 }

```

So the derived subgroup is normal.. This is always true of the derived subgroup of any group (though in a simple group (in the technical sense) it consists only of the identity.)

The above list of subgroups is complete. As an exercise, sketch the lattice diagram to show how these fit inside S_3 .

Advanced Option: Use PermGroups in the same way as WordGroups to determine the character table of S_3 .

3.4 S_4 and its subgroups A_4 and D_4

With PermGroups D_4 can be generated using $N = 4$ digits with a pair of seed permutations (1234) and (13), or else the pair (1234) and (12)(34). Recall that you enter the latter as 1234 and 12'34 respectively. Once generated, follow the analysis of Part 2, §2.5.

One way to generate A_4 is with $N = 4$ digits, and the two seed permutations (123) and (12)(34). One way to generate S_4 is with $N = 4$ digits, and the two seed permutations (123) and (14).

Follow the corresponding analysis in Part 2. In determining subgroups, note that there must be four subgroups of type S_3 within S_4 , each corresponding to one of the digits 1, 2, 3 or 4 being constant.

Advanced option : We will use this section to illustrate the importance of the permutation representation and permutation character, and the operation of 'restriction' of a character from a group G to its subgroup H .

Because each of these groups, S_4 , A_4 , D_4 , is generated by combining permutations on 4 digits, each element can be represented by a 4th order permutation matrix in which the entries are either 0 or 1, there being exactly one 1 in each row and in each column. For example, the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

represents the permutation $(12) = (12)(3)(4)$ because if it operates by multiplication on the column vector $(a_1 \ a_2 \ a_3 \ a_4)^T$, it gives the column vector $(a_2 \ a_1 \ a_3 \ a_4)^T$ (T means transpose.). The trace of this matrix is 2, so the character value for the conjugacy class containing (12) in the permutation character is 2. In general, the character value is the number of digits which remain fixed by the permutation; here it is 2, being digits 3 and 4.

If you run PermGroups for the groups S4, A4 and D4 you will find the permutation characters noted below.

S4: the character table has the form

Class label	K1	K3	K5	K2	K4
Element id no.	1	3	14	2	5
Element	1	14	12'34	123	1423
Number in class	1	6	3	8	6
Order of element	1	2	2	3	4
Order centraliser	24	4	8	3	4
<hr/>					
Trivial char.	1	1	1	1	1
	1	-1	1	1	-1
	2				
	3				
	3				

and, by inspection of the representative elements heading the conjugacy classes, the permutation character is $(4 \ 2 \ 0 \ 1 \ 0)$. Take the dot product of this with itself, weighted by the number of elements in each class to get $16 + 24 + 0 + 8 + 0 = 48 = 2|G|$, so there must be two irreducible characters making up the permutation character. If you subtract the trivial character from $(4 \ 2 \ 0 \ 1 \ 0)$ you get $(3 \ 1 \ -1 \ 0 \ -1)$. The weighted dot product of this with itself is $9 + 6 + 3 + 0 + 6 = 24$, so $(3 \ 1 \ -1 \ 0 \ -1)$ must be one of the irreducible characters.

A4: The framework of the character table is found to be

Class label	K1	K3	K2	K4
Element id no.	1	3	2	4
Element	1	12'34	123	132
Number in class	1	3	4	4
Order of element	1	2	3	3
Order centraliser	12	4	3	3
<hr/>				
Trivial char.	1	1	1	1
	1	1	e	e*
	1	1	e*	e
	3			

$e = \exp(2\pi i/3)$

The permutation character is, by inspection on row 3 in the header, $(4 \ 0 \ 1 \ 1)$. Its weighted dot product with itself is $16 + 0 + 4 + 4 = 24 = 2|G|$, so there are two irreducible characters added to make $(4 \ 0 \ 1 \ 1)$. Subtract the trivial character to get $(3 \ -1 \ 0 \ 0)$ which has the weighted self dot product $9 + 3 = |G|$, so the missing degree 3 character must be $(3 \ -1 \ 0 \ 0)$.

D4: Create the group from the seed permutations (1234) and (13) . The partial character table is:

Class label	K1	K3	K4	K5	K2
Element id no.	1	3	4	5	2
Element	1	13	13'24	14'23	1234
Number in class	1	2	1	2	2
Order of element	1	2	2	2	4
Order centraliser	8	4	8	4	4
<hr/>					
Trivial char.	1	1	1	1	1
	1	-1	1	-1	1
	1	1	1	-1	-1
	1	-1	1	1	-1
	2				

With PermGroups you can show that the character induced from the normal subgroup $C4$ is $(2 \ 0 \ -2 \ 0 \ 0)$. The permutation character is read from the header of the above table as $(4 \ 2 \ 0 \ 0 \ 0)$ and its weighted self dot product is $16 + 8 = 3|G|$, so it is comprised of three irreducible characters. Subtract the sum of Rows 1 and 3 from $(4 \ 2 \ 0 \ 0 \ 0)$ to get $(2 \ 0 \ -2 \ 0 \ 0)$ and check that this has self dot product $4 + 0 + 4 = |G|$. Again the permutation character has led easily to a non-linear irreducible character.

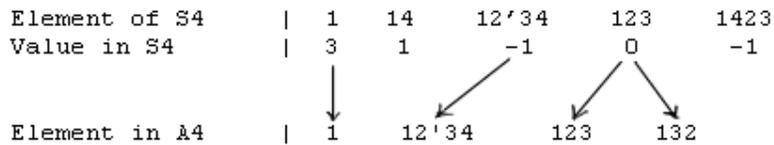
Now 'restriction'. Because $D4$ is a subgroup of $S4$, a 4×4 permutation matrix representing an element g in $D4$ must also be a matrix in $S4$. This matrix must have the same trace *i.e.* character value, whether considered as an element of $S4$ or of $D4$. Therefore, if β is the character value for the conjugacy class in $S4$ which contains g , β must also be the character value of g and its conjugates within the table for $D4$. We can therefore derive a character of $D4$ from the table for $S4$ by picking out those conjugacy classes of $S4$ which appear in $D4$.

Here is an example of restricting the character $(3 \ 1 \ -1 \ 0 \ -1)$ of $S4$ to $D4$. From the headers of the $S4$ and $D4$ tables:

Element of $S4$		1	14	12'34	123	1423
Value in $S4$		3	1	-1	0	-1
		↓	↓	↓	↘	↓
Element in $D4$		1	13	13'24	14'23	1234

Now match the elements in $D4$ with those in $S4$ using the structure of the permutations as shown by the arrows above. $D4$ does not have any order 3 elements so the character value of (123) does not transfer to $D4$. The restricted character of $D4$ is $(3 \ 1 \ -1 \ -1 \ -1)$. This is the sum of rows 3 and 5 in the $D4$ table: $(1 \ 1 \ 1 \ -1 \ -1) + (2 \ 0 \ -2 \ 0 \ 0) = (3 \ 1 \ -1 \ -1 \ -1)$.

The equivalent operation from $S4$ to $A4$ is shown below and yields the character $(3 \ -1 \ 0 \ 0)$, as found above.



3.5 A non-abelian direct product $S_3 \times C_3$.

Run PermGroups, specify $N = 6$ digits, 1 to 6, and enter the two seed permutations 123456 and 12'34'56. A group of 18 elements is created. This must be a subgroup of S_6 . The program finds

```

1 element of order 1 : 1
6 elements of order 6 : 2 10 11 12 13 14
3 elements of order 2 : 3 15 16
8 elements of order 3 : 4 5 6 7 8 9 17 18

```

These fall into 9 conjugacy classes.

In this exercise we will first try to determine the subgroup structure. The list of subgroups found automatically by the program is

```

Subgroup No. 2 of 9 elements : 1 4 5 6 7 8 9 17 18
Subgroup No. 3 of 6 elements : 1 2 4 5 14 15
Subgroup No. 4 of 6 elements : 1 3 4 5 10 11
Subgroup No. 5 of 6 elements : 1 4 5 12 13 16
Subgroup No. 6 of 3 elements : 1 4 5 (= Group centre, Z)
Subgroup No. 7 of 3 elements : 1 17 18 (= derived/commutator subgroup)

```

- *Subgroup no. 2:* Since there are no elements of order 9, this must be the direct product $C_3 \times C_3$. Indeed it is easy to show, using the program's multiply elements facility, that it is the Cartesian product of the centre Z and the derived subgroup, *i.e.* element sets $\{1, 4, 5\} \times \{1, 17, 18\}$.
- *Subgroups 3, 4, and 5:* Similarly, you can show that each of the 6-element subgroups above is a direct product $C_2 \times C_3$. Subgroup number 3 is the Cartesian product of sets $\{1, 15\} \times \{1, 4, 5\}$, subgroup 4 is $\{1, 3\} \times \{1, 4, 5\}$, and subgroup 5 is $\{1, 16\} \times \{1, 4, 5\}$.
- *Subgroup no. 7:* The derived subgroup is the permutations set $\{1, 153'246, 135'264\}$.

The program does not find all subgroups automatically, so we have to search for other possible candidates. This can be done by trial and error when the program asks "Do you wish to enter (from the keyboard) another subgroup which you have identified?". If you input any set of trial elements, the program will check for inconsistencies and warn you of missing elements.

As an example, let us try to find another subgroup with six elements by augmenting the derived subgroup. Since the derived subgroup has elements of order 3, we are looking to form its product with an element of order 2. So try element number 3. Enter order of subgroup is 6, then element numbers 1, 17, 18, 3, 3, 3. (The repeating of 3 does not matter.) The program responds that element 15 is missing, being numbers 17×3 . So now enter 1, 17, 18, 3, 15, 3. It responds that element 16 is missing. If you now type 1, 17, 18, 3, 15, 16, the input is accepted as forming a subgroup, new to the list, and added as Subgroup No. 8.

Clearly $\{1, 17, 18\}$ is a subgroup of this new group. By examining the products of elements, you can show that it is the symmetric group S_3 . If you select to examine its cosets, you will see that

it is normal in G. This is consistent with the whole group being $S_3 \times C_3 = \{ 1, 3, 15, 16, 17, 18 \} \times \{ 1, 4, 5 \}$.

I leave it as an exercise to the reader to draw a lattice diagram showing the nesting of these subgroups.

Advanced option: The 6 cosets of the commutator (derived) subgroup are listed as follows :

Coset 1 is { (1) (153'246) (135'264) }	= elements No. { 1 17 18 }
Coset 2 is { (123456) (163254) (143652) }	= elements No. { 2 11 12 }
Coset 3 is { (12'34'56) (16'23'45) (14'25'36) }	= elements No. { 3 16 15 }
Coset 4 is { (135'246) (264) (153) }	= elements No. { 4 9 7 }
Coset 5 is { (153'264) (135) (246) }	= elements No. { 5 6 8 }
Coset 6 is { (145236) (125634) (165432) }	= elements No. { 10 13 14 }

You have to decide what group these form, as six composite elements. Note that element 2 has order 6. You can use the facility to calculate products of elements to show that successive powers of element 2 are respectively elements 2, 4, 15, 5, 14, 1 = identity. These appear in the cosets numbered 2, 4, 3, 5, 6, 1 respectively. If you choose another element of order 6 in coset 2, say number 11, you can calculate its powers to be elements 11, 4, 3, 5, 10, 1 which lie in coset numbers 2, 4, 3, 5, 6, 1, exactly as for element 2. We can conclude that the abelianisation factor group is C_6 , the cyclic group on six elements. Hence when the program asks 'do the cosets for a cyclic group of order 6?' enter 'y' then 2 for the element which generates the subgroup.

When the partially complete character table is printed, it has six completed linear characters, lifted from the abelianisation factor group. There are three characters of degree 2 yet to find. To find a character of order 2 we should induce on a subgroup of order $|G|/2 = 9$. Thus choose Subgroup No. 2, which is $C_3 \times C_3 = \{1, 4, 5\} \times \{1, 17, 18\}$. There are nine characters in the table of $C_3 \times C_3$ which one could choose. First choose the character in which element no. 4 has value 1 and no. 17 has value c , where c is $\exp(2\pi i/3)$. Accordingly, when invited to enter the character value for each element, enter No. 1 is 1, 4 is 1, 5 is 1, 6 is c , 7 is c^* , 8 is c^* , 9 is c , 17 is c and 18 is c^* . The induced character is reported as

$$(2 \quad 0 \quad 2 \quad 2 \quad c+c^* \quad c+c^* \quad c+c^* \quad 0 \quad 0)$$

which is $(2 \quad 0 \quad 2 \quad 2 \quad -1 \quad -1 \quad -1 \quad 0 \quad 0)$ when the value for c is substituted. This is a new irreducible character, as can be verified by showing that its dot product with itself, weighted by the number of elements in each conjugacy class, is $18 = |G|$. Note that the matrix representation corresponding to this character cannot be faithful because the matrix for permutation (123456) is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{which has order 2, not 6.}$$

When this character is entered into the character table, we have only two characters yet to find. Suspecting that these are complex conjugates of each other, I have entered the unknowns p to w shown on the next page. $e = \exp(2\pi i / 6) = \exp(\pi i / 3)$. We now try to solve for these.

Recall that the permutation character has values for each conjugacy class equal to the number of digits left fixed by the permutations in that class. For instance, element 6 = (135) moves 3 digits and leaves 3 fixed, so has character value 3. Refer to the header of the character table and the permutation elements listed with the cosets of the derived subgroup above. You can write down the permutation character as $(6 \ 0 \ 0 \ 0 \ 3 \ 3 \ 0 \ 0 \ 0)$. The weighted dot product of this with itself is $36 + 18 + 18 = 72 = 4|G|$ so it is comprised of four irreducible characters. I leave it to you to confirm that this is the sum of rows 1, 3, 8 and 9.

The dot product of columns 1 and 2 (K1 with K3) gives $p+p^* = 0$ so $p = 0$. The dot product of columns 1 and 7 (K1 with K9) gives $6 - 2 + 2(u + u^*) = 0$, so $u + u^* = -2$. This suggests that $u = -1$, a conjecture confirmed by taking the dot product of column K9 with itself : $7 + 2uu^* = 9$ which is the order of the centraliser.

The dot product of column 8 (class K2) with itself gives $6 + 0 + 2vv^* = 6$, since $|e| = 1$. We conclude that $v = 0$. Similarly $w = 0$.

Group $S_3 \times C_3$: partial table

Class label	K1	K3	K4	K5	K6	K7	K9	K2	K8
Element id no.	1	3	4	5	6	7	17	2	10
Number in class	1	3	1	1	2	2	2	3	3
Order of element	1	2	3	3	3	3	3	6	6
Order centraliser	18	6	18	18	9	9	9	6	6
<hr/>									
Trivial char.	1	1	1	1	1	1	1	1	1
	1	-1	e2	e2*	e2*	e2	1	e	e*
	1	1	e2*	e2	e2	e2*	1	e2	e2*
	1	-1	1	1	1	1	1	-1	-1
	1	1	e2	e2*	e2*	e2	1	e2*	e2
	1	-1	e2*	e2	e2	e2*	1	e*	e
	2	0	2	2	-1	-1	-1	0	0
	2	p	q	r	s	t	u	v	w
	2	p*	q*	r*	s*	t*	u*	v*	w*

To find q , first take the dot product of columns 1 and 3 (K1, K4). Since $e^2 + e^{2*} = -1$, $q+q^* = -2$, or $2\text{Re}[q] = -2$. Next take the weighted dot product of column K4 with itself to get $6 + 4 + 2qq^* = 18$, or $qq^* = 4$. Together these imply that $q = -1 \pm i\sqrt{3} = 2e^2$ or $2e^{2*}$. We are free to choose $q = 2e^2$.

From comparing columns 3 and 4, r must equal q or q^* . To decide which, dot the two columns together and remember that we dot one column with the complex conjugate of the other : $2 + 2(e^4 + e^{4*}) + 4 + qr^* + q^*r = 0$. Since $e^4 + e^{4*} = -1$, $qr^* + (qr^*)^* = -4$, which implies that $\text{Re}[qr^*] = -2$. Now qq^* is positive definite, so we conclude that $r^* = q$. Indeed $q^2 = -2 \pm 2i\sqrt{3}$.

To determine s and t take the respective column dot products with i) themselves, ii) with column 1 and iii) with each other. i) gives $ss^* = tt^* = 1$. ii) gives $s + s^* = t + t^* = 1$. Together with i) these give $s = (1 \pm i\sqrt{3})/2$, $t = (1 \pm i\sqrt{3})/2 = e$ or e^* . iii) gives $st^* + (st^*)^* = -1$ so $s = t^*$.

Finally we have to decide whether $s = e$ or e^* . So take the (complex) dot products of the columns containing q and s . On substituting $2e^2$ for q this gives $e^2s^* + (e^2s^*)^* = 1$. If $s = e^*$, the left side of this equation would be -2 , whereas $s = e$ gives $e + e^* = 1$.

From all the above, the last two characters in the table are

Row 8		2	0	2e2	2e2*	e	e*	-1	0	0
Row 9		2	0	2e2*	2e2	e*	e	-1	0	0

Confirmation of these comes from inducing a character from the subgroup no. 3, the centre Z. If you select to induce the character $(1 \ c \ c^*)$, $c = \exp(2\pi i/3) = e^2$, in row 2 of the corresponding C3 character table, you get the character of $G = S_3 \times C_3$:

$$6 \quad 0 \quad 6c \quad 6c^* \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad .$$

This can be identified as the sum of rows (twice row 2 plus row 5 plus row 8) in the character table. Similarly, if you induce from the subgroup no. 3 = { 1 2 4 5 14 15 }, and use the character in the character in row 2 of the C6 table, you obtain the following character of G:

$$3 \quad -1 \quad 3e^2 \quad 3e^{2*} \quad 0 \quad 0 \quad 0 \quad c \quad c^*$$

This is the sum of rows 2 and 8.

Note that the matrices representing elements 2 and 3 induced from this subgroup are

$$(123457) = \begin{bmatrix} e & 0 & 0 \\ & & \\ 0 & 0 & e^2 \\ & & \\ 0 & 1 & 0 \end{bmatrix} \quad (12)(34)(56) = \begin{bmatrix} 0 & 1 & 0 \\ & & \\ 1 & 0 & 0 \\ & & \\ 0 & 0 & -1 \end{bmatrix}$$

These appear to give a faithful representation of G since the order of the first matrix is 6 and of the second is 2, in agreement with the respective permutations. I leave it to you the reader to decide whether this is the lowest dimension faithful representation of $S_3 \times C_3$:

3.6 The Symmetric Group S5 and Alternating Group A5

This section looks at larger groups not included in Part 2 on WordGroups. A central aim of the exercise is to examine the subgroups of A5 and show by direct evaluation that it has no normal subgroups. Hence it cannot be the direct product of two constituent groups, nor even a semi-direct product. It cannot be at the head of a composition series. Because of this it is termed 'simple'.

The parent symmetric group S5 has 120 elements. It can be generated by specifying N = 5 digits, 1 to 5, and two seed permutations 12345 and 12. It is clear that no element can have order greater than 5, so no subgroup can be the direct product of cyclic groups of the form $C_m \times C_n$ where m and n are coprime, since this would be isomorphic to C_{mn} , $mn > 5$. The factors of 120 are 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30, 40 and 60, so the only possibilities for direct products of cyclic groups are $C_2 \times C_2$, $C_2 \times C_2 \times C_2$, $C_2 \times C_2 \times C_2 \times C_2$, $C_2 \times C_4$, $C_4 \times C_4$, and $C_2 \times C_2 \times C_4$. (Direct products of non-cyclic groups might still be allowed provided there were no element of order >5.)

A5 can be generated by specifying N = 5 digits, 1 to 5, and two seed permutations 12345 and 12'34. A5 is the subgroup of S5 formed from an even number of transpositions, so any elements of order 2 must be 'double', of the form (12)(34); single transposition elements such as (12) do not exist in A5.

A5 has 60 elements. The factors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20 and 30, so the only possibility for a direct product subgroup is $C_2 \times C_2$. There are only two groups of order 6, C6 and S3, so any subgroup of order 6 must be S3. Any subgroup of order 10 must be the dihedral D5, which is equivalent to the semi-direct product $C_5 \rtimes C_2$. Any subgroup of order 12 must be A4 since the other order-12 groups, $C_2 \times C_6$, D6 and the dicyclic Q6, all have elements of order 6. By the same argument any subgroup of order 20 must be the semi-direct product $C_5 \rtimes C_4$. There can be no subgroups of order 15 or 30. In summary, the possibilities for A5's subgroups are

C2
 C3
 C4, C2×C2.
 C5
 S3
 D5 = C5 ∧ C2
 A4
 C5 ∧ C4.

The program lists the orders of all elements

```

1 element of order 1 : 1
24 elements of order 5 : 2 4 5 10 11 12 13 14 15
16 18 19 22 23 24 25 39 40 44 45 48 49 53 54
15 elements of order 2 : 3 17 26 27 32 35 36 43 50 55
56 57 58 59 60
20 elements of order 3 : 6 7 8 9 20 21 28 29 30 31
33 34 37 38 41 42 46 47 51 52
  
```

There are no elements of order 4 so we can eliminate C4 and C5 ∧ C4 from the above list of possible subgroups. PermGroups finds the following subgroups from the centralisers, centre Z and derived subgroup. I have noted the type of each on the right.

```

Subgroup No. 1 of 5 elements : 1 2 4 5 16 C5
Subgroup No. 2 of 5 elements : 1 10 11 44 45 C5
Subgroup No. 3 of 5 elements : 1 12 13 39 40 C5
Subgroup No. 4 of 5 elements : 1 14 15 24 25 C5
Subgroup No. 5 of 5 elements : 1 18 19 48 49 C5
Subgroup No. 6 of 5 elements : 1 22 23 53 54 C5
Subgroup No. 7 of 4 elements : 1 3 50 57 C2×C2
Subgroup No. 8 of 4 elements : 1 17 59 60 C2×C2
Subgroup No. 9 of 4 elements : 1 26 43 55 C2×C2
Subgroup No. 10 of 4 elements : 1 27 36 58 C2×C2
Subgroup No. 11 of 4 elements : 1 32 35 56 C2×C2
Subgroup No. 12 of 3 elements : 1 6 7 C3
Subgroup No. 13 of 3 elements : 1 8 9 C3
Subgroup No. 14 of 3 elements : 1 20 21 C3
Subgroup No. 15 of 3 elements : 1 28 29 C3
Subgroup No. 16 of 3 elements : 1 30 31 C3
Subgroup No. 17 of 3 elements : 1 33 34 C3
Subgroup No. 18 of 3 elements : 1 37 38 C3
Subgroup No. 19 of 3 elements : 1 41 42 C3
Subgroup No. 20 of 3 elements : 1 46 47 C3
Subgroup No. 21 of 3 elements : 1 51 52 C3
Subgroup No. 22 of 1 elements : 1 = centre, Z I
Subgroup No. 23 of 60 elements : whole group = derived subgroup
  
```

One striking feature of S5 and A5 is that they have only five conjugacy classes for so many elements. Another feature of A5 is that the derived (commutator) subgroup is the whole group! No other group examined in this document has this property. Recall that the commutator elements are as far from being commutative as is possible, and the fact that no element other than the identity commutes with any other is also shown by Z being trivial (subgroup number 22).

PermGroups has not detected any subgroup A4, D5 or S3 so we must look for these ourselves. It will be clear that A4 will be formed when one of the five digits 1, 2, 3, 4 or 5 is held constant. Thus we can expect five A4 subgroups. By selecting permutations which do not contain digit 1, 2, 3, 4 or 5 respectively, we can enter from the keyboard five A4 subgroups with these elements:

```

Digit 1 fixed: 1 8 9 17 30 31 33 34 41 42 59 60
Digit 2 fixed: 1 6 7 28 29 32 35 41 42 51 52 56
Digit 3 fixed: 1 8 9 20 21 27 36 37 38 51 52 58
Digit 4 fixed: 1 6 7 26 33 34 37 38 43 46 47 55
Digit 5 fixed: 1 3 20 21 28 29 30 31 46 47 50 57 .

```

A similar argument leads to the view that there must be ten S3 subgroups, corresponding to the 5C_3 combinations of the digits 1 to 5. Here are three of them:

```

From digits 1,2,3 : 1 17 36 46 47 56
From digits 1,4,5 : 1 17 26 51 52 57
From digits 2,3,4 : 1 26 30 31 35 58 .

```

The D5 subgroups must involve powers of an element of order 5 and an element of order 2. For instance, powers of element 2 are elements {1, 2, 4, 5, 16}; try combinations of this set with the elements of order 2. Since there are 24 elements of order 5, and four appear in each D5, we might expect $24/4 = 6$ D5 subgroups. I list these below, and have separated the order 5 elements (first set) from the order 2 ones (second set).

```

Subgroup D5a : 1 { 2 4 5 16 } (55 56 57 58 59)
Subgroup D5b : 1 {10 11 44 45} {17 27 35 50 55}
Subgroup D5c : 1 {12 13 39 40} (26 32 36 50 59)
Subgroup D5d : 1 {18 19 48 49} { 3 26 27 56 60}
Subgroup D5e : 1 {22 23 53 54} { 3 17 32 43 58}
Subgroup D5f : 1 {14 15 24 25} {35 36 43 57 60}.

```

Note that each order 2 element appears in two D5 subgroups.

Thought it is rather tedious, you can enter all these subgroups from the keyboard and use PermGroups to find their cosets. You will find that in every case the right cosets differ set-wise from the left ones, demonstrating that A5 has no normal subgroups. Algebraic proof of this fact for A5 and all higher alternating groups is given in textbooks.

Advanced option: The program prints the framework of the character table as .

Group A5					
Class label	K1	K3	K5	K2	K4
Element id no.	1	3	6	2	4
Element	1	12'34	135	12345	13524
Number in class	1	15	20	12	12
Order of element	1	2	3	5	5
Order centraliser	60	4	3	5	5
<hr/>					
Trivial char.	1	1	1	1	1
		3			
		3			
		4			
		5			

From the header, the permutation character is readily seen to be $(5 \ 1 \ 2 \ 0 \ 0)$. This must be faithful, but is not necessarily irreducible. In fact the weighted dot product of this with itself is $25 + 15 + 80 = 120 = 2|G|$, so we look for two irreducible characters within it. Subtract the trivial character to get $(4 \ 0 \ 1 \ -1 \ -1)$ and note that its weighted self dot product is $16 + 20 + 12 + 12 = |G|$, so this must be the irreducible character of dimension 4.

To find the other characters we have to induce from subgroups. Choose the A4 subgroup with digit 5 held constant; this has elements { 1, 3, 20, 21, 28, 29, 30, 31, 46, 47, 50, 57}. These have to be matched to conjugacy classes in A4 so that each element can be given the correct character value. If you induce from the trivial character of an A4 subgroup, you just get the permutation character again. Let's choose to induce from row 3 of the A4 character table. Here is the relevant extract from the A4 table:

Element	1	12'34	123	132	
Order of element	1	2	3	3	
	1	1	t*	t	t = exp(2πi/3)

The order-1 and -2 elements in the A4 subgroup must be allocated value 1. These are elements {1, 3, 50, 57}. The order-3 elements are all in one conjugacy class in A5, but split into two classes in A4, one being the inverse of the other. You need to identify the elements in these two classes in A4 and find which elements these permutations correspond to in A5. Thus element (123) is in the same class as (134), (243) and (142) and so they all have character value t*. Enter the appropriate character values into PermGroups; it finds matrix representations for the two seeded permutations:

$$\begin{array}{l}
 \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ [& & & &] \\ 1 & 0 & 0 & 0 & 0 \\ [& & & &] \\ 0 & 1 & 0 & 0 & 0 \\ [& & & &] \\ 0 & 0 & 1 & 0 & 0 \\ [& & & &] \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\
 (12345) =
 \end{array}
 \qquad
 \begin{array}{l}
 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ [& & & &] \\ 0 & 0 & t^* & 0 & 0 \\ [& & & &] \\ 0 & t & 0 & 0 & 0 \\ [& & & &] \\ 0 & 0 & 0 & 0 & t \\ [& & & &] \\ 0 & 0 & 0 & t^* & 0 \end{bmatrix} \\
 (12)(34) =
 \end{array}$$

where $t = \exp(2\pi i/3)$. The corresponding induced character is

5	1	t^*+t	0	0
5	1	-1	0	0

which evaluates to

Using matrix multiplication you can show that this representation is faithful. Also the weighted dot product of this character with (4 0 1 -1 -1) is $20 - 20 = 0$, so we gain confidence that rows 4 and 5 of the character table are correct.

It is tempting to suspect that the two characters of order 3 are complex conjugates, or share some similar symmetry, but for the time being let us eschew this and treat the eight undetermined character elements as independent. The character table looks as follows:

Group A5 : partial table

Class label	K1	K3	K5	K2	K4
Element id no.	1	3	6	2	4
Element	1	12'34	135	12345	13524
Number in class	1	15	20	12	12
Order of element	1	2	3	5	5
Order centraliser	60	4	3	5	5
Trivial char.	1	1	1	1	1
	3	p	q	r	s
	3	t	u	v	w

4	0	1	-1	-1
5	1	-1	0	0

Take the weighted complex dot product of rows 1 and 2, and rows 4 and 2:

$$\begin{aligned}
 3 + 15p + 20q + 12r + 12s &= 0 \\
 \frac{12 + 0p + 20q - 12r - 12s}{15 + 15p + 40q} &= 0 \text{ so}
 \end{aligned}$$

Compare this with the dot product of rows 5 and 2

$$15 + 15p - 20q = 0$$

and we have that $q = 0, p = -1$. By the same argument $u = 0, t = -1$.

We find confirmation of these values from the dot product of column 2 with itself, which should equal the order of the centraliser for class K3. We find $1 + pp^* + tt^* + 1 = 4$ or $pp^* + tt^* = 1$, consistent with $p = t = -1$. Also take the dot product of columns 1 and 2 to obtain $p + t = -2$. With $q = u = 0$ the dot products of columns 1, 2, 3 with themselves and with each other are consistent. The degree 3 characters now look like

$$\begin{array}{ccccc}
 3 & -1 & 0 & r & s \\
 3 & -1 & 0 & v & w
 \end{array}$$

The determination of the remaining values is more challenging, considering that they are probably complex numbers. By taking the weighted dot products of rows 1 with 2, and 1 with 3 we find

$$r + s = 1, \quad v + w = 1.$$

Dot products of columns 1 with 4, 1 with 5 give

$$r + v = 1, \quad s + w = 1.$$

The simultaneous solution of these four equations is $r = w, s = v = 1 - r$. The character table now looks as follows:

Group A5					
Class label	K1	K3	K5	K2	K4
Element id no.	1	3	6	2	4
Element	1	12'34	135	12345	13524
Number in class	1	15	20	12	12
Order of element	1	2	3	5	5
Order centraliser	60	4	3	5	5
	1	1	1	1	1
	3	-1	0	r	1-r
	3	-1	0	1-r	r
	4	0	1	-1	-1
	5	1	-1	0	0

This satisfies all the orthogonality criteria. As a further check, recall that the number of times an irreducible character contributes to the regular character is given by its value in the first column of the table. We can check the table against the regular character as follows

1	1	1	1	1
9	-3	0	3r	3-3r
9	-3	0	3-3r	3r
16	0	4	-4	4
25	5	-5	0	0
60	0	0	0	0

So far, so good, but we do not yet know whether r is real or complex. Take the dot product of one of the above degree-3 characters with itself to obtain

$$9 + 15 + 12 rr^* + 12(1-r)(1-r^*) = |G|,$$

giving $2rr^* - r - r^* = 2$.

- a) If r is real, $r^2 - r - 1 = 0$, with solutions $r = (1 \pm \sqrt{5})/2$, so if $r = (1 + \sqrt{5})/2$, $1 - r = (1 - \sqrt{5})/2$, the algebraic conjugate. We see a symmetry between the two degree-3 characters.
- b) If r is complex, $r = \alpha + i\beta$, we have $\alpha^2 - \alpha + \beta^2 = 1$. Moreover, by a general theorem of characters, if r is complex, the two degree-3 characters must be complex conjugates of each other. So $\alpha - i\beta = 1 - \alpha - i\beta$, implying that $\alpha = 1/2$. and $\beta^2 = 5/4$. So it seems that r could be $(1 + i\sqrt{5})/2$ and $1 - r = r^*$, another symmetrical situation.

So which of these possibilities is correct?

It seems necessary to induce some further characters from subgroups. Using PermGroups to induce from the $C2 \times C2$, $C3$ and $C5$ centraliser subgroups, I find these reducible characters:

a)	15	-1	0	0	0	4 G
b)	20	0	-1	0	0	7 G
c)	12	0	0	$c+c^*$	c^2+c^{2*}	where $c = \exp(2\pi i/5)$ 3 G
d)	12	0	0	c^2+c^{2*}	c^*+c	3 G

I have noted to the right the weighted dot product of each character with itself, showing the number of irreducible characters of which it is the sum. a) is the sum of character rows 2 to 5. b) is c) plus row 5 (though this is only 5 characters, not 7 – perhaps the reader can explain this?)

In c) and d) note that $c+c^* = 0.618\dots = (-1 + \sqrt{5})/2$ and $c^2+c^{2*} = -1.618\dots = -(1 + \sqrt{5})/2$.

c) could be the sum of rows 2, 4 and 5 = $(12 \ 0 \ 0 \ r-1 \ -r)$. This is consistent with $r = (1 + \sqrt{5})/2$, $1 - r = (1 - \sqrt{5})/2$ but not with $r - 1$ and r being complex.. We may conclude that option a) above is correct, that r is real and

$$r = (1 + \sqrt{5})/2, \quad 1 - r = (1 - \sqrt{5})/2$$

This completes the A5 character table.

Here's a challenge question : is A5 the smallest group with only one linear character? If so, why?

John M Coffey
Cheshire, UK
January 2009, with additions January 2017.

APPENDIX

Table of orders and one relation $\langle ba = \text{'word'} \rangle$ which generate finite groups using WordGroups

This is a table of pairs of generators **a** and **b** (with orders given in the first and second columns respectively) which *may* form a finite group when used with relation **ba** = word (in third column). Note that semi-direct products of cyclic groups $C_m = \langle a \rangle$ and $C_n = \langle b \rangle$ are formed according to a relation $ba = a^k . b$ where k and m are coprime. Many of the groups indicated below are therefore semi-direct products.

2 2 ab	3 2 ab	4 5 ab
	3 2 $a^2.b$.	4 5 $a.b^2$.
2 3 ab		4 5 $a.b^3$.
2 3 $a.b^2$.	3 3 ab	4 5 $a.b^4$.
2 4 ab	3 4 ab	4 6 ab
2 4 $a.b^3$.	3 4 $a^2.b$.	4 6 $a.b^5$.
		4 6 $a^3.b$.
2 5 ab	3 5 ab	
2 5 $a.b^4$.		4 7 ab
		4 7 $a.b^6$.
2 6 ab	3 6 ab	
2 6 $a.b^5$.	3 6 $a^2.b$.	
		4 8 ab
2 7 ab	3 7 ab	4 8 $a.b^3$.
2 7 $a.b^6$.	3 7 $a.b^2$.	4 8 $a.b^5$.
	3 7 $a.b^4$.	4 8 $a.b^7$.
2 8 ab	3 8 ab	4 9 ab
2 8 $a.b^3$.	3 8 $a^2.b$.	4 9 $a.b^8$.
2 8 $a.b^5$.		
2 8 $a.b^7$.		
	3 9 ab	4 10 ab
2 9 ab	3 9 $a.b^4$.	4 10 $a.b^3$.
2 9 $a.b^8$.	3 9 $a.b^7$.	4 10 $a.b^7$.
		4 10 $a.b^9$.
	3 10 ab	
2 10 ab	3 10 $a^2.b$.	4 11 ab
2 10 $a.b^9$.		4 11 $a.b^{10}$.
2 11 ab	3 11 ab	4 12 ab
2 11 $a.b^{10}$.		4 12 $a.b^5$.
		4 12 $a.b^7$.
2 12 ab	3 13 ab	4 12 $a.b^{11}$.
2 12 $a.b^5$.	3 13 $a.b^3$.	
2 12 $a.b^7$.	3 13 $a.b^9$.	4 13 ab
2 12 $a.b^{11}$.		4 13 $a.b^5$.
		4 13 $a.b^8$.
2 13 ab	3 14 ab	4 13 $a.b^{12}$.
2 13 $a.b^{12}$.	3 14 $a.b^9$.	
	3 14 $a.b^{11}$.	
		4 14 ab
2 14 ab	3 15 ab	4 14 $a.b^{13}$.
2 14 $a.b^{13}$.		
	3 16 ab	4 15 ab
2 15 ab		4 15 $a.b^2$.
2 15 $a.b^4$.		4 15 $a.b^4$.
2 15 $a.b^{11}$.		4 15 $a.b^7$.
2 15 $a.b^{14}$.		4 15 $a.b^8$.
		4 15 $a.b^{11}$.
		4 15 $a.b^{13}$.
		4 15 $a.b^{14}$.
2 16 ab	4 2 ab	
2 16 $a.b^7$.	4 2 $a^3.b$.	4 16 ab
2 16 $a.b^9$.		4 16 $a.b^3$.
2 16 $a.b^{15}$.	4 3 ab	4 16 $a.b^5$.
	4 3 $a.b^2$.	4 16 $a.b^7$.
	4 4 ab	
	4 4 $a.b^3$.	
	4 4 $a^3.b$.	

4 16 $a.b^9.$
4 16 $a.b^{11}.$
4 16 $a.b^{13}.$

5 2 ab
5 2 $a^4.b.$

5 3 ab

5 4 ab
5 4 $a^2.b.$
5 4 $a^3.b.$
5 4 $a^4.b.$

5 5 ab

5 6 ab
5 6 $a^4.b.$

5 7 ab

5 8 ab
5 8 $a^2.b.$

5 9 ab

5 10 ab

5 11 ab
5 11 $a.b^3.$
5 11 $a.b^4.$
5 11 $a.b^5.$
5 11 $a.b^9.$

5 12 ab

5 13 ab

5 14 ab

5 15 ab

5 16 ab

6 2 ab
6 2 $a^5.b.$

6 3 ab
6 3 $a.b^2.$

6 4 ab
6 4 $a.b^3.$
6 4 $a^5.b.$

6 5 ab
6 5 $a.b^4.$

6 6 ab
6 6 $a.b^5.$

6 7 ab
6 7 $a.b^2.$
6 7 $a.b^3.$
6 7 $a.b^4.$
6 7 $a.b^5.$
6 7 $a.b^6.$

6 8 ab
6 8 $a.b^3.$
6 8 $a.b^5.$
6 8 $a.b^7.$

6 9 ab
6 9 $a.b^2.$
6 9 $a.b^4.$
6 9 $a.b^5.$
6 9 $a.b^7.$
6 9 $a.b^8.$

6 10 ab
6 10 $a.b^9.$

6 11 ab
6 11 $a.b^{10}.$

6 12 ab
6 12 $a.b^5.$
6 12 $a.b^7.$
6 12 $a.b^{11}.$

6 13 ab
6 13 $a.b^3.$
6 13 $a.b^4.$
6 13 $a.b^9.$
6 13 $a.b^{10}.$
6 13 $a.b^{12}.$

6 14 ab
6 14 $a.b^3.$
6 14 $a.b^5.$
6 14 $a.b^9.$
6 14 $a.b^{11}.$
6 14 $a.b^{13}.$

6 15 ab
6 15 $a.b^4.$
6 15 $a.b^{11}.$
6 15 $a.b^{14}.$

6 16 ab
6 16 $a.b^7.$
6 16 $a.b^9.$

7 2 ab
7 2 $a^6.b.$

7 3 ab
7 3 $a^2.b.$
7 3 $a^4.b.$

7 4 ab
7 4 $a^6.b.$

7 5 ab

7 6 ab
7 6 $a^2.b.$
7 6 $a^3.b.$

7 7 ab

7 8 ab

7 9 ab
7 9 $a^2.b.$

7 10 ab

7 11 ab

7 12 ab

7 13 ab

7 14 ab

7 15 ab

7 16 ab

8 2 ab
8 2 $a^3.b.$
8 2 $a^5.b.$
8 2 $a^7.b.$

8 3 ab
8 3 $a.b^2.$

8 4 ab
8 4 $a.b^3.$
8 4 $a^3.b.$
8 4 $a^5.b.$
8 4 $a^7.b.$

8 5 ab
8 5 $a.b^2.$
8 5 $a.b^3.$
8 5 $a.b^4.$

8 6 ab
8 6 $a.b^5.$
8 6 $a^3.b.$

8 7 ab
8 7 $a.b^6.$

8 8 ab
 8 8 a.b³.
 8 8 a.b⁵.
 8 8 a.b⁷.

 8 9 ab
 8 9 a.b⁸.

 8 10 ab
 8 10 a.b³.
 8 10 a.b⁷.
 8 10 a.b⁹.

 8 11 ab
 8 11 a.b¹⁰.

 8 12 ab
 8 12 a.b⁵.
 8 12 a.b⁷.
 8 12 a.b¹¹.

 8 13 ab
 8 13 a.b⁵.
 8 13 a.b⁸.
 8 13 a.b¹².

 8 14 ab
 8 14 a.b¹³.

 8 15 ab
 8 15 a.b².
 8 15 a.b⁴.
 8 15 a.b⁷.
 8 15 a.b⁸.
 8 15 a.b¹¹.
 8 15 a.b¹³.
 8 15 a.b¹⁴.

 8 16 ab
 8 16 a.b³.
 8 16 a.b⁵.
 8 16 a.b⁷.
 8 16 a.b⁹.
 8 16 a.b¹¹.
 8 16 a.b¹³.

9 2 ab
 9 2 a⁸.b.

 9 3 ab
 9 3 a⁴.b.
 9 3 a⁷.b.

 9 4 ab
 9 4 a⁸.b.

 9 5 ab

9 6 ab
 9 6 a².b.

 9 7 ab
 9 7 a.b².
 9 7 a.b⁴.

 9 8 ab

 9 9 ab
 9 9 a.b⁴.
 9 9 a.b⁷.

 9 10 ab

 9 11 ab

 9 12 ab

 9 13 ab
 9 13 a.b³.
 9 13 a.b⁹.

 9 14 ab
 9 14 a.b⁹.
 9 14 a.b¹¹.

 9 15 ab

 9 16 ab

10 2 ab
 10 2 a⁹.b.

 10 3 ab
 10 3 a.b².

 10 4 ab
 10 4 a.b³.
 10 4 a³.b.
 10 4 a⁷.b.
 10 4 a⁹.b.

 10 5 ab
 10 5 a.b⁴.

 10 6 ab
 10 6 a.b⁵.

 10 7 ab
 10 7 a.b⁶.

 10 8 ab
 10 8 a.b³.
 10 8 a.b⁵.
 10 8 a.b⁷.

 10 9 ab
 10 9 a.b⁸.

10 10 ab
 10 10 a.b⁹.

 10 11 ab
 10 11 a.b².
 10 11 a.b³.
 10 11 a.b⁴.
 10 11 a.b⁵.
 10 11 a.b⁶.
 10 11 a.b⁷.
 10 11 a.b⁸.
 10 11 a.b⁹.
 10 11 a.b¹⁰.

 10 12 ab
 10 12 a.b⁵.
 10 12 a.b⁷.
 10 12 a.b¹¹.

 10 13 ab
 10 13 a.b¹².

 10 14 ab
 10 14 a.b¹³.

 10 15 ab
 10 15 a.b⁴.
 10 15 a.b¹¹.
 10 15 a.b¹⁴.

 10 16 ab
 10 16 a.b⁷.
 10 16 a.b⁹.

11 2 ab
 11 2 a¹⁰.b.

 11 3 ab

 11 4 ab
 11 4 a¹⁰.b.

 11 5 ab
 11 5 a³.b.
 11 5 a⁴.b.
 11 5 a⁵.b.

 11 6 ab

 11 7 ab

 11 8 ab

 11 9 ab

 11 10 ab

 11 11 ab

11 12 ab
11 13 ab
11 14 ab
11 15 ab
11 16 ab

12 2 ab
12 2 $a^5.b$.
12 2 $a^7.b$.
12 2 $a^{11}.b$.

12 3 ab
12 3 $a.b^2$.

12 4 ab
12 4 $a.b^3$.
12 4 $a^5.b$.
12 4 $a^7.b$.
12 4 $a^{11}.b$.

12 5 ab
12 5 $a.b^2$.
12 5 $a.b^3$.
12 5 $a.b^4$.

12 6 ab
12 6 $a.b^5$.

12 7 ab
12 7 $a.b^2$.
12 7 $a.b^3$.
12 7 $a.b^4$.
12 7 $a.b^5$.
12 7 $a.b^6$.

12 8 ab
12 8 $a.b^3$.
12 8 $a.b^5$.
12 8 $a.b^7$.

12 9 ab
12 9 $a.b^2$.
12 9 $a.b^4$.
12 9 $a.b^5$.
12 9 $a.b^7$.
12 9 $a.b^8$.

12 10 ab
12 10 $a.b^3$.
12 10 $a.b^7$.
12 10 $a.b^9$.

12 11 ab
12 11 $a.b^{10}$.

12 12 ab
12 12 $a.b^5$.
12 12 $a.b^7$.
12 12 $a.b^{11}$.

12 13 ab
12 13 $a.b^2$.
12 13 $a.b^3$.
12 13 $a.b^4$.
12 13 $a.b^5$.
12 13 $a.b^6$.
12 13 $a.b^7$.
12 13 $a.b^8$.
12 13 $a.b^9$.
12 13 $a.b^{10}$.
12 13 $a.b^{11}$.
12 13 $a.b^{12}$.

12 14 ab
12 14 $a.b^3$.
12 14 $a.b^5$.
12 14 $a.b^9$.
12 14 $a.b^{11}$.
12 14 $a.b^{13}$.

12 15 ab
12 15 $a.b^2$.
12 15 $a.b^4$.
12 15 $a.b^7$.
12 15 $a.b^8$.
12 15 $a.b^{11}$.
12 15 $a.b^{13}$.
12 15 $a.b^{14}$.

12 16 ab
12 16 $a.b^3$.
12 16 $a.b^5$.
12 16 $a.b^7$.
12 16 $a.b^9$.
12 16 $a.b^{11}$.
12 16 $a.b^{13}$.

13 2 ab
13 2 $a^{12}.b$.

13 3 ab
13 3 $a^3.b$.
13 3 $a^9.b$.

13 4 ab
13 4 $a^5.b$.
13 4 $a^8.b$.
13 4 $a^{12}.b$.

13 5 ab

13 6 ab
13 6 $a^3.b$.

13 7 ab

13 8 ab

13 9 ab

13 10 ab

13 11 ab

13 12 ab

13 13 ab

13 14 ab

13 15 ab

13 16 ab

14 2 ab
14 2 $a^{13}.b$.

14 3 ab
14 3 $a.b^2$.
14 3 $a^9.b$.
14 3 $a^{11}.b$.

14 4 ab
14 4 $a.b^3$.

14 5 ab
14 5 $a.b^4$.

14 6 ab
14 6 $a.b^5$.
14 6 $a^3.b$.

14 7 ab
14 7 $a.b^6$.

14 8 ab
14 8 $a.b^3$.
14 8 $a.b^5$.
14 8 $a.b^7$.

14 9 ab
14 9 $a.b^8$.

14 10 ab
14 10 $a.b^9$.

14 11 ab
14 11 $a.b^{10}$.

14 12 ab
14 12 $a.b^5$.
14 12 $a.b^7$.
14 12 $a.b^{11}$.

14 13 ab
14 13 a.b¹².

14 14 ab
14 14 a.b¹³.

14 15 ab
14 15 a.b⁴.
14 15 a.b¹¹.
14 15 a.b¹⁴.

14 16 ab
14 16 a.b⁷.
14 16 a.b⁹.

15 13 ab
15 13 a.b³.
15 13 a.b⁹.

15 14 ab
15 14 a.b⁹.
15 14 a.b¹¹.

15 15 ab

15 16 ab

16 11 ab
16 11 a.b¹⁰.

16 12 ab
16 12 a.b⁵.
16 12 a.b⁷.
16 12 a.b¹¹.

16 13 ab
16 13 a.b⁵.
16 13 a.b⁸.
16 13 a.b¹².

15 2 ab
15 2 a⁴.b.
15 2 a¹¹.b.
15 2 a¹⁴.b.

15 3 ab

15 4 ab
15 4 a².b.
15 4 a⁴.b.
15 4 a⁷.b.
15 4 a⁸.b.
15 4 a¹¹.b.

15 5 ab

15 6 ab

15 7 ab
15 7 a.b².
15 7 a.b⁴.

15 8 ab
15 8 a².b.

15 9 ab
15 9 a.b⁴.
15 9 a.b⁷.

15 10 ab

15 11 ab
15 11 a.b³.
15 11 a.b⁴.
15 11 a.b⁵.
15 11 a.b⁹.

15 12 ab

16 2 ab
16 2 a⁷.b.
16 2 a⁹.b.
16 2 a¹⁵.b.

16 3 ab
16 3 a.b².

16 4 ab
16 4 a.b³.
16 4 a³.b.
16 4 a⁵.b.
16 4 a⁷.b.
16 4 a⁹.b.
16 4 a¹¹.b.

16 5 ab
16 5 a.b².
16 5 a.b³.
16 5 a.b⁴.

16 6 ab
16 6 a.b⁵.

16 7 ab
16 7 a.b⁶.

16 8 ab
16 8 a.b³.
16 8 a.b⁵.
16 8 a.b⁷.

16 9 ab
16 9 a.b⁸.

16 10 ab
16 10 a.b³.
16 10 a.b⁷.
16 10 a.b⁹.

16 14 ab
16 14 a.b¹³.

16 15 ab
16 15 a.b².
16 15 a.b⁴.
16 15 a.b⁷.
16 15 a.b⁸.
16 15 a.b¹¹.
16 15 a.b¹³.
16 15 a.b¹⁴.

16 16 ab
16 16 a.b³.
16 16 a.b⁵.
16 16 a.b⁷.
16 16 a.b⁹.
16 16 a.b¹¹.
16 16 a.b¹³.

(END of LIST)