Fitting a curve to 3 or 4 points: some options explored

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1 Context and statement of the problem

In developing a suite of computer programs to simulate the generation of simple pieces of music I have had a need to find mathematical curves along which the shape of each musical phrase can be guided. Typical curve shapes are shown in Figure 1 which is taken from the principal article on *www.mathstudio.co.uk*. We are concerned here only with the contours made from smooth curves, labelled types 4+ to 7−. For each such phrases the first and last notes are given in semitones on a scale of pitch (the MIDI scale), and either one peak or one trough is given (Case 1), or else a peak and a trough are selected in both time and pitch (Case 2). The issue is to find mathematical expressions which will fit the 3 or 4 given points and have smooth behaviour otherwise.

The two Cases can be stated by reference to the notation in Figure 2. The two panels represent two separate phrases. The musical contour in the left panel will start at a given time and pitch (t_1, p_1) and close at (t_2, p_2). At some time t_a between these, chosen at random, an intermediate pitch p_a is introduced, also chosen at random. Call this position A. The requirement is that this be either a peak or a trough, and that the three points be joined by a smooth curve, like either of the two dotted curves in the left panel. Case 1 will create a guiding contour of types 4+, 4−, 5+, 5−, 6+, 6−. The right panel shows Case 2. There are two randomly selected intermediate points A = (t_a, p_a), B = (t_b, p_b) with $p_a > p_1$ and $p_b < p_2$ (type 7+), or its mirror image.

Your first thought may be that this is a simple problem – curve fitting is very well established. Let me explain why I think the general case is not straightforward. Take Case 1, the left panel in Figure 2. There are 3 given points, so a parabola should go precisely through them because it has 3 disposable coefficients. Yes, a parabola can be constructed through these 3 points, but it will not in general have its peak at (t_a, p_a). The requirement to have a peak is an extra constraint so 4 adjustable coefficients are required. So do we work with a cubic curve $p = a + bt + ct^2 + dt^3$? It is possible to solve the four simultaneous equations which will fit a cubic through the 3 points and have a peak at (t_a, p_a), but in the generality of instances there will be an associated trough too, and this may be so deep that the curve is musically unacceptable. A cubic is acceptable only over a limited range of parameters. So we might consider other curves with 4 coefficients, including the rational functions

$$\frac{a + bt + ct^2}{1 + dt}, \qquad \frac{a + bt + ct^3}{1 + dt}, \qquad \frac{a + bt^2 + ct^3}{1 + dt}, \qquad \frac{1 + at^2}{b + ct + dt^2}, \qquad etc.$$

and hope that the simultaneous equations linking the coefficients can be solved. One might alternatively consider for Case 1 curves variations on transcendental functions based on $x^2 \exp(-x)$ or

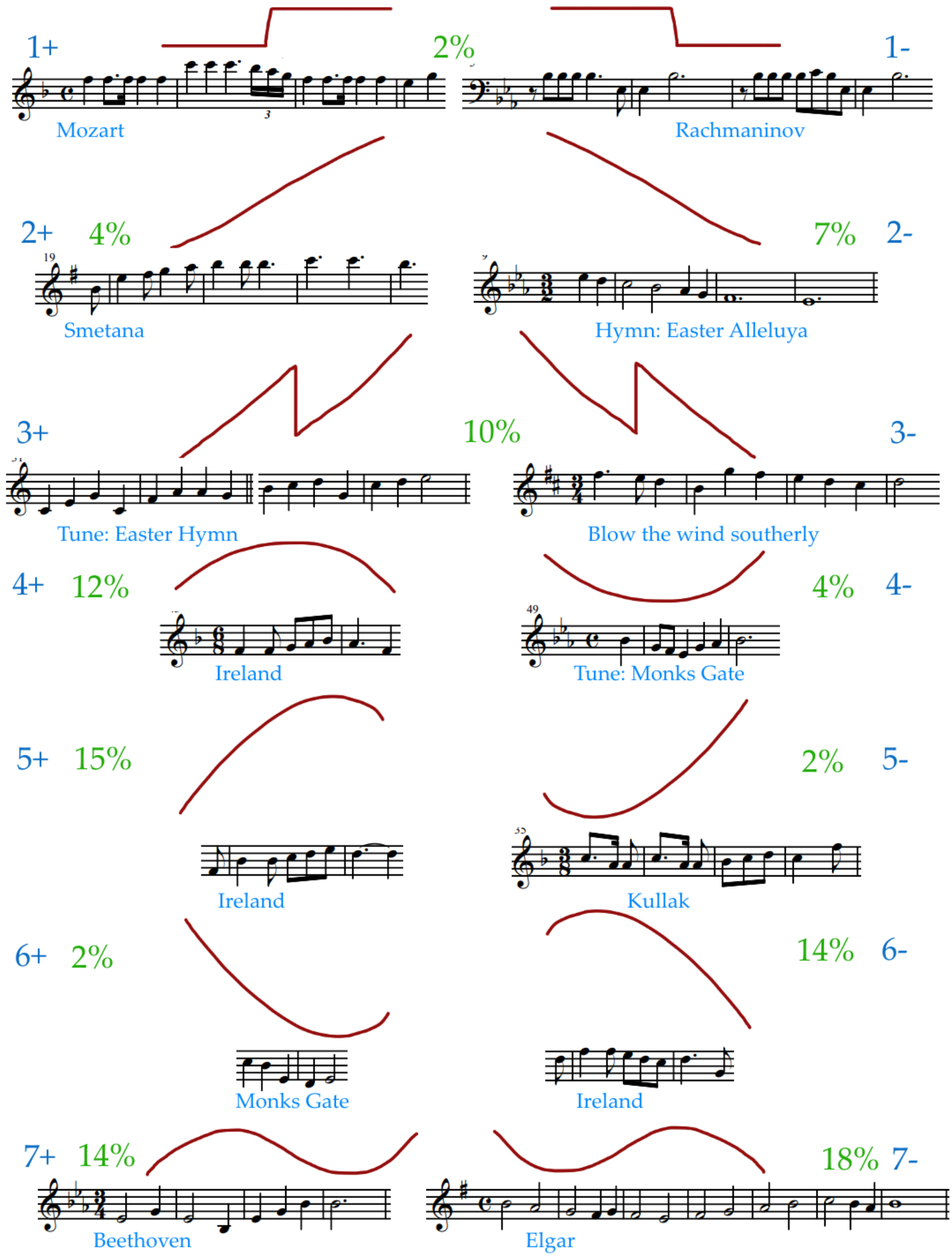

Figure 1: Examples of musical phrases and their shapes.

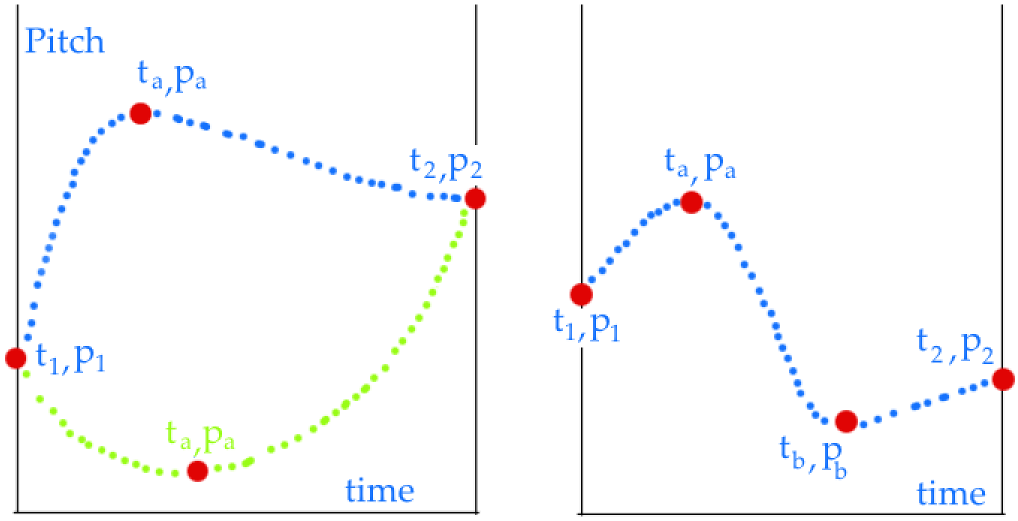

Figure 2: Left: Case 1. Fit to 3 points where the central one is a maximum or a minimum. Right: Case 2. Fit to 4 points where the middle two are peak and trough, or trough and peak.

$x^2 \exp(-x^2)$ or others. Yet another approach would be make a crude approximation by a set of linked straight lines. Or perhaps some piecewise stitching of cubic splines. Or linked Bézier curves? This article looks into some of these possibilities.

2 Useful scope of a cubic

Start with a general cubic and find its first and second derivatives:

$$p = a + bt + ct^2 + dt^3, \qquad \frac{dp}{dt} = b + 2ct + 3dt^2, \qquad \frac{d^2p}{dt^2} = 2c + 6dt.$$

There is one point of inflection at $\frac{d^2p}{dt^2} = 0$, at $t = -c/3d$. The curve is symmetrical about this point. If $d > 0$, that is the t^3 term is positive, the curve draws up from bottom left to top right. Suppose we require that A be a peak, as with the blue curve in the left panel of Figure 2. For a peak to form the gradient at the point of inflection, $b - c^2/3d$, must be negative. When a peak occurs, there must necessarily be a trough mirrored in the point of inflection. If the trough lies outside the phrase's time boundaries, the created curve is Case 1, types 4+ to 6−. If on the other hand the trough lies at t_c so that $t_1 < t_c < t_a$ or $t_a < t_c < t_2$, we have Case 2, types 7+ or 7−, though of course without control of the trough's position or pitch. So a cubic could in principle form all the eight types of phrase curve from 4+ to 7− in Figure 1.

I find from some numerical experiments that the cubic is satisfactory only so long as point A is sufficiently close to the mid point of the phrase and not too far in pitch from p_1 and p_2. If A is near the start of the interval, a high curvature is required there in order for the curve to bend down to (t_1, p_1). The cubic then oscillates with an exaggerated ~ shape and the trough is too far below p_2 to be musically meaningful.

Here are the values of the coefficients to create a peak or trough at A $= (t_a, p_a)$:

$$d = \frac{p_1(t_2 - t_a)^2 - p_2(t_1 - t_a)^2 + p_a(t_1 - t_2)(t_1 + t_2 - 2t_a)}{(t_1 - t_2)(t_1 - t_a)^2(t_2 - t_a)^2}$$

$$c = \frac{p_1 - p_a - d t_1^3 + 3 d t_1 t_a^2 - 2 d t_a^3}{(t_1 - t_a)^2}$$

$$b \;=\; -2 c t_a - 3 d t_a^2, \qquad a = p_1 - c t_1^2 + 2 c t_1 t_a - d t_1^3 + 3 d t_1 t_a^2 \,. \tag{1}$$

Figure 3 gives several examples of satisfactory and unsatisfactory cubics. In all figures $t_1 = 0$, $p_1 = 1$, $t_2 = 10$, $p_2 = 5$. In the left panel the peak at A is at $p_a = 6$ and t_a steps 4, 5, 6, 7. All of these curves would be musically acceptable, though the blue one is type 7+, the others type 5+. The trough appears when the peak is placed too close to the low starting note at $t_1 = 0$. In the central panel the red and blue curves might be acceptable, but not the green because it oscillates too violently. Peak point A here was selected at (6, 3) (red), (2, 4) (green) and (2, 3) (blue). In the right hand panel A has been selected near the end of the phrase at $t_a = 9$ (red) and the others at $t_a = 8$. Only the green curve is type 5+. The blue might be acceptable, but not the black or red.

The evidence is that a cubic would work well for type 4+, 4− curves and some type 5, 6 and 7 ones provided the single selected peak point A is fairly near the centre of the phrase and not too high in pitch above the higher end point. Of course, in selecting a peak or trough, the user has no control over the position of the attendant trough (or *vice verse*). In practice the software would have to be given limits of acceptability and reject curves which fell outside. It is easy to determine the position and pitch of the attendant trough by mirroring in the point of inflection at $t = -c/3d$.

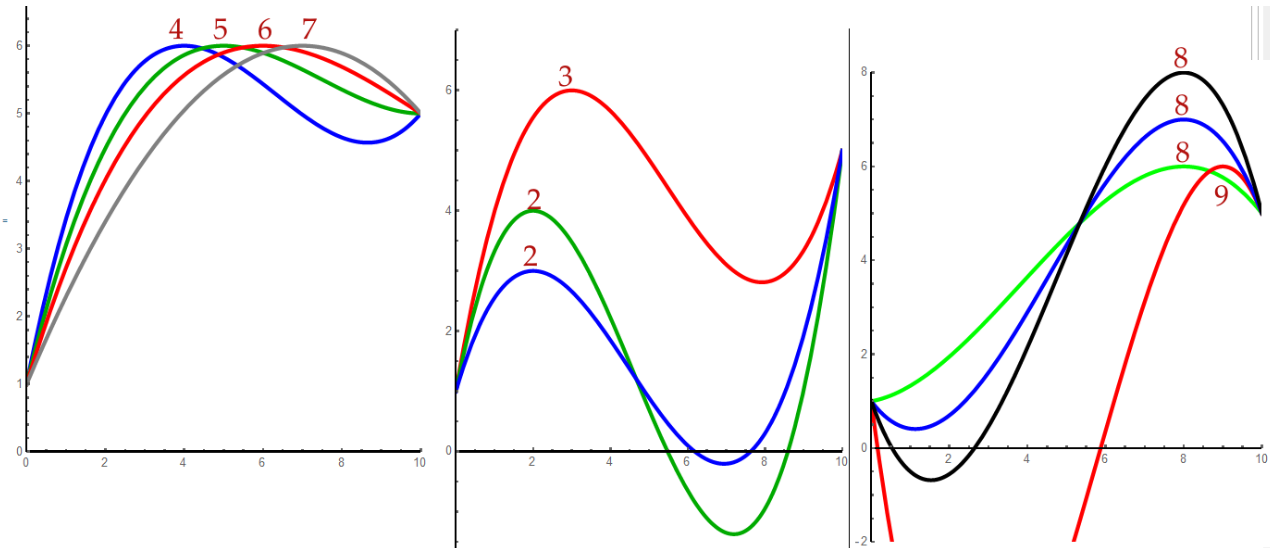

Figure 3: Examples of fitting a single cubic function to three given points with a peak at the middle point. The t co-ordinate of each peak A is noted on the curve.

2.1 A rational function

The motivation is to find a function with 4 parameters which will allow a single peak or trough near either end point of the phrase. The function

$$p \;=\; \frac{a + bt + ct^3}{t - d}$$

has two branches, one either side of the singularity at $t = d$, and one of these has the shape we need. This curve is called 'Newton's Trident'. Figure 4 shows the case $p = -(1 + t + t^3)/t$ and we want the

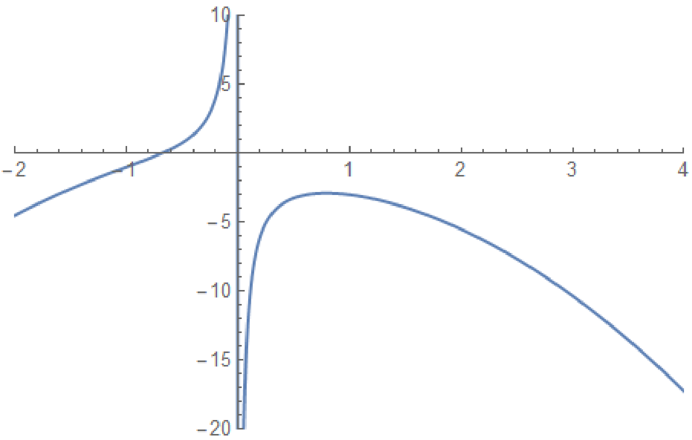

Figure 4: The function $p = -(1+t+t^3)/t$, a model for 'trident' curves.

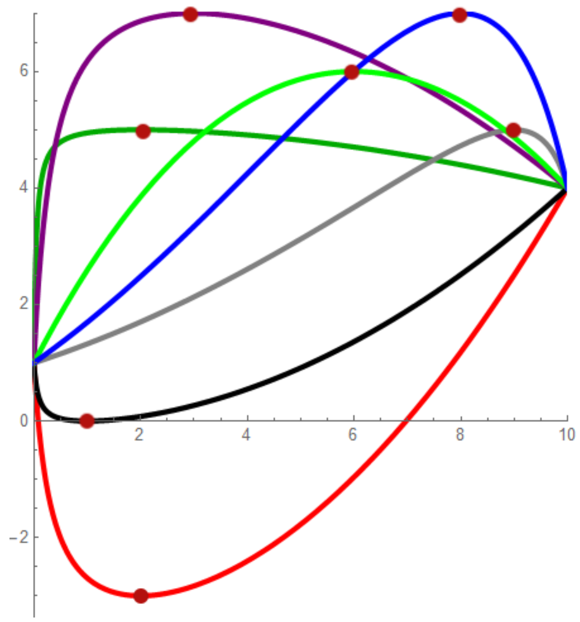

Figure 5: Seven examples of trident curves fitted to the two end points. The required positions of the five peaks and two troughs are marked by red spots.

region of the right of the y axis with its single peak. The formulae which fit the curve at the two end points and give a single peak at A are:

$$a \;=\; p_a(t_a - d) - bt_a - c{t_a}^3, \qquad b \;= p_a - 3c\,{t_a}^2 \tag{2}$$

$$c \;=\; \frac{(p_1 - p_a)(t_1 - d)}{(t_1 - t_a)^2(t_1 + 2t_a)}$$

$$d \;=\; \frac{p_1 t_1 (t_2 - t_a)^2 (t_2 + 2t_a) - p_2 t_2 (t_1 + 2t_a)(t_1 - t_a)^2 + p_a\left[t_1^3 t_2 - t_1\left(t_2^3 + 2t_a^3\right) + 2t_2 t_a^3\right]}{p_1 (t_2 - t_a)^2 (t_2 + 2t_a) - p_2 (t_1 + 2t_a)(t_1 - t_a)^2 + p_a (t_1 - t_2)\left(t_1^2 + t_1 t_2 + t_2^2 - 3t_a^2\right)}$$

Seven examples are plotted in Figure 5. The end points of the phrase are (0, 1) and (10, 4). Perhaps the dark green curve rises too steeply near the left margin and its peak is too flat, but otherwise these results are all satisfactory. Because there is no possibility of a spurious trough or unwanted peak, this function may be superior to cubics for Case 1 curves.

2.2 Piecewise straight lines and Bézier curves.

Approximation by straight line segments, though crude in itself, will pave the way to describing Bézier curves. Straight lines can be used for both Case 1 (one peak or trough) or Case 2 (one peak, one trough). The idea is to represent a peak or trough by a horizontal line segment, p = constant. Each end is then connected to the end points or to the other horizontal segment as in the left panel of Figure 6. This is certainly not smooth, but it is a robust construction which will work in all cases. There is a choice to be made of how long each horizontal segment should be, and here only judgement can be used. If the width in time of the phrase is 10 units, as in our examples so far, perhaps $\pm 1{\cdot}5$ of the peak is reasonable for a single point A near the centre of the phrase. However, where the peak is near the start or end of the phrase, or close to a trough in Case 2, these horizontal lines should not encroach on the space of the sloping line segments, making them too steep. A rule of thumb might be that each horizontal segment should not extend more than a quarter of the way in t to the next joint, in either direction. It terms of practical application to musical phrases, this stick model might be a good enough.

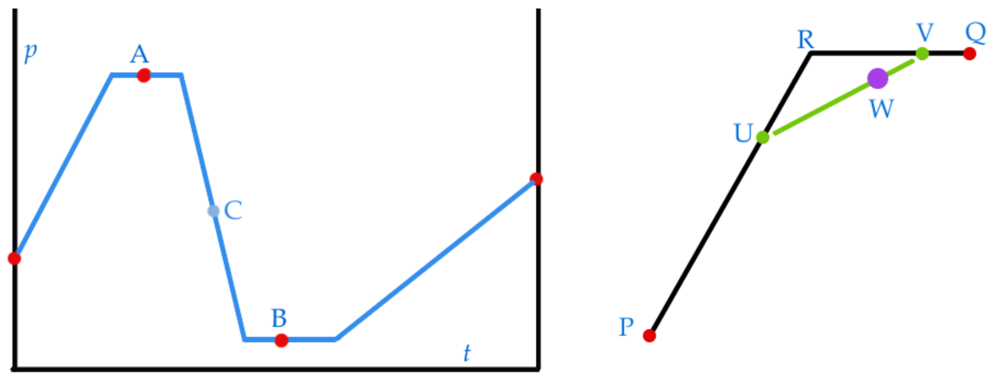

Figure 6: Left: Approximation to a peak A and trough B by straight lines segments. Right: Development of line segments into controls of a Bézier curve traced by point W.

The purpose of extending to Bézier curves is to round off the corners where adjacent line segments meet. The first step is to regard each pair of adjacent line segments as tangents to a curved segment which joins their end points. Consider the general case illustrated in the right panel of Figure 6. Two points P and Q are given initially, and R is added so as to define the required gradients at P and Q. A general point U on the line PR is parametrised by the variable u, $0 \leq u \leq 1$, such that

$$U_x = P_x + u(R_x - P_x) = (1-u)P_x + uR_x, \qquad U_y = (1-u)P_y + uR_y$$

where the subscript denotes the horizontal or vertical component. Similarly

$$V_x = R_x + u(Q_x - R_x) = (1-u)R_x + uQ_x, \qquad V_y = (1-u)R_y + uQ_y.$$

When $u = 0$, U is at P, V is at R, and when $u = 1$, U is at R, V is at Q. Now join UV with a third straight line and parametrise the general point W by

$$W_x = U_x + u(V_x - U_x) = (1-u)U_x + uV_x, \qquad W_y = (1-u)U_y + uV_y. \tag{3}$$

As u increases from 0 to 1, W will trace out a smooth curve from P to Q which is tangent to PR and to RQ at its endpoints. The curve will lie inside the triangle PQR.

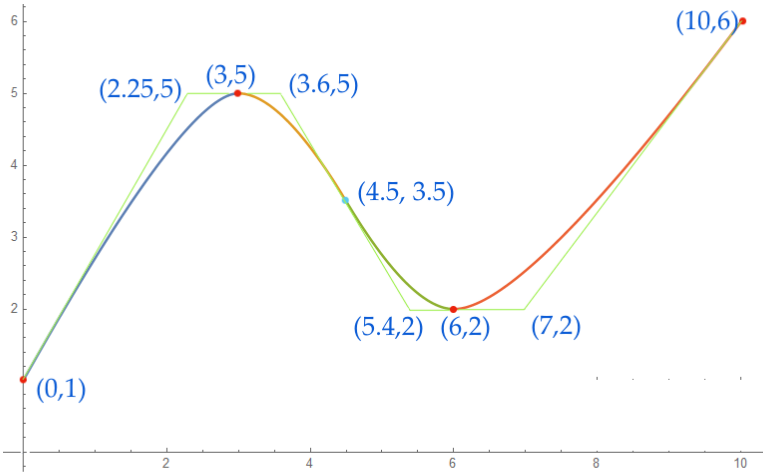

Figure 7: Example of 4 linked Bézier curves creating a peak and trough at selected positions.

A set of such Bézier curves can be constructed to form more or less any smooth curve through given points with given gradients. Referring to the left panel of Figure 6, to join the peak A to the trough B it is necessary is introduce an intermediate point C near the middle of the sloping straight line which joins them. Two curves are fitted, one between A and C, the other from C to B. The tangents to the two curves at C will coincide, so ensuring that there is no angular there. Discontinuities in the second derivatives cannot be avoided in this scheme, but for the purpose in hand this does not matter at all.

An example is shown in Figure 7. The given points are the start at (0,1), the end at (10,6), a peak at (3,5) and a trough at (6,2). The auxiliary points marked in the figure create the horizontal sections through the peaks, shown in pale green. They extend 25% of the way in t to the next corner. The intermediate point C at (4·5, 3·5) is a point of inflection. The whole curve looks exactly what is wanted. The method can equally well apply to Case 1 curves which have only the one peak or trough.

2.3 A cubic spline

The last scheme is also to create a Case 2, phrase types 7+, 7− peak and trough curve. The idea is to introduce an intermediate point C between peak and trough, and fit one cubic on its left through the peak at A and a separate cubic on its right through the trough at B, the two cubics coinciding at C. C can be taken as half way down the straight line joining A and B. With the points as in Figure 7 this would be at (4·5, 3·5) again. We have from §2 the formulae to fit the unique cubic through start, A and C, and another through C, B and end. The result is shown in the left panel of Figure 8. The right panel is for somewhat different positions of peak and trough. The results are similar to the Bézier curves above and are wholly satisfactory. Provided the peak and trough are not too close to start or end or too high or low, we are not troubled with the spurious associated trough and peak.

2.4 Application to musical phrases

This study has led to four schemes for creating Case 1 and Case 2 curves. I consider all are satisfactory except for a single cubic in those cases where it produces a highly oscillating curve with unwanted associated trough (or peak). In the article on *www.mathstudio.co.uk* called 'Some simple computer

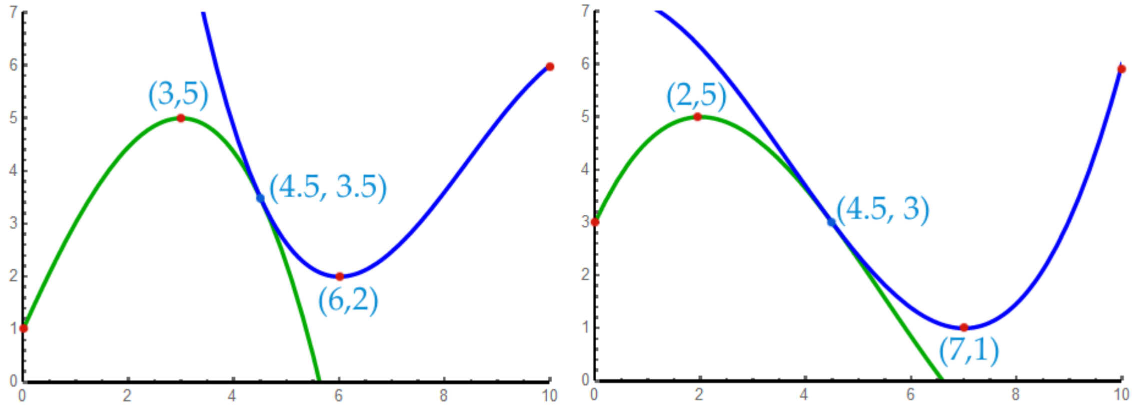

Figure 8: Two examples of a pair of cubic curves spliced together either side of a point on the line joining peak to trough.

programs to simulate construction of tonal music' I present some statistical evidence for the typical duration of phrases and the pitch range they occupy. Broadly, a phrase lasts for two bars in 2-, 3- or 4-time, meaning that in 4-time one lasts for 8 crotchet beats. Typically the harmony will change between 2 and 4 times in a phrase, the average being $3\frac{1}{3}$. Harmonic progressions are simple: often I - V, I - IV - I, I - IV - V, IV - V - I, I - IV - V - I. The pitch difference between first and last notes can be in the range from 0 semitones (i.e. it starts and ends on the same note) to an octave. The most common intervals are unison (0 semitones), perfect 4th (5 semitones), fifth (7) and octave (12). The relative frequency of phrase shapes is noted in Figure 1. The pitch interval between highest and lowest notes is on average about 8 semitones, with the most common values being the perfect 4th (5 semitones), perfect 5th (7), minor 7th (10) and octave (12).

This information is enough to simulate musical phrases. For each phrase a guide curve is created according to this prescription and laid on top of a matrix of notes. There is one column of notes per beat, and the notes filling each column are all those within the triad of that beat. Then at each beat the user selects the note in the prevailing triad on the row which lies closest to the guide curve. This selection of crotchet notes gives the framework to the melody. It can be decorated to taste with passing and auxiliary notes, suspensions and appoggiaturas in a rhythm to taste. Four examples are given in Figure 9. The guide curve is sketched in the top stave with the notes at blue spots where the curve intersects the prevailing triad (marked by a chord letter, C, G or F major). Some harmony notes have been added in the bass stave to complete the harmony. The reader will recognise the phrase types as 5+, 4+, 7− and 6−.

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Figure 9: Four examples of how musical phrases can be derived from a background harmony overlaid with a guide curve, and decorated with passing and auxiliary notes.