Solutions to exercises

Chapter 1

1.7 $x > 16.25$.

1.9 $x < -3$ or >2.

1.18 (i) $-2 < x < 0$. (ii) $x < 0$.

1.34 1. (i) $-2.5 < x < -2$. (ii) $x < -4$ or > -2. (iii) $-1.5 < x < -1$. (iv) $1.5 < x < 4$.

6. Observe that $M - \varepsilon$ cannot be an upper bound.

Chapter 2

2.2 (i) Take $N \geqslant 1/\sqrt{\varepsilon}$. (ii) Take $N \geqslant \log_2 1/\varepsilon$. (iii) Take $N \geqslant 1/\varepsilon^2$.

2.13 (i) 1. (ii) 0. (iii) -1.

2.21 (i) Take $N \geqslant \sqrt{C}$. (ii) Take $N \geqslant C^2$. (iii) Take $N \geqslant \log_2 C$.

2.32 (i) Not monotonic. (ii) Is monotonic (strictly increasing).

2.39 1. (i) Converges to 1. (ii) Diverges to infinity. (iii) Diverges to infinity. Use Bernoulli's inequality and the open sandwich principle. (iv) Converges to 1. Use Bernoulli and the sandwich principle. (v) Sequence is $\leqslant (\frac{5}{6})^n$ for all $n \geqslant 3$, therefore is null. (vi) Is null. Divide through by 4^n. (vii) Diverges to infinity. (viii) Oscillates. (ix) Oscillates.

3. Last part. Write $\sqrt{(n+1)} - \sqrt{n} = 1/[\sqrt{(n+1)} + \sqrt{n}]$.

5. The converse requires $a_n > 0$.

9. For each positive integer n there exists $a_n \in E$ such that $a_n > n$.

10. Choose a subsequence (a_{n_r}) which converges to a. (a_n) cannot converge to a so there must exist $\varepsilon > 0$ such that for any N we must have $|a_n - a| \geqslant \varepsilon$ for some $n > N$. This enables us to choose a subsequence (a_{m_r}) such that $|a_{m_r} - a| \geqslant \varepsilon$ for all r. Now choose a convergent subsequence of (a_{m_r}). It must converge to $b \neq a$.

11. $a = \sqrt{A}$.

12. $l = \sqrt{2}$.

Chapter 3

3.3 $\sum_1^\infty 1/n(n+1) = 1$.

3.23 (i) Convergent. (ii) Divergent.

3.29 (i) Conditionally convergent. (ii) Absolutely convergent.

3.32 (i) Converges absolutely for all x. (ii) Converges absolutely for $|x| < 1$, conditionally for $x = \pm 1$, diverges for $|x| > 1$.

3.34 (i) $\frac{1}{3}$. (ii) $\frac{1}{4}$.

3.38 1. They all converge except (ii) and (vi).

2. (i) Absolutely convergent. (ii) Conditionally convergent. (iii) Divergent. (iv) Conditionally convergent. (v) Conditionally convergent. (vi) Absolutely convergent.

3. (i) $\frac{3}{2}$. (ii) 1. (iii) 1. (iv) 4/27. (v) 1/e. (vi) 2.

5. Limit of the sequence is 0. Radius of convergence of the power series is 2.

6. The series is divergent. Observe, e.g., that

$$\frac{1}{\sqrt{(2n-1)}} - \frac{1}{2n} \geqslant \frac{1}{2\sqrt{(2n-1)}}$$

for all $n \geqslant 1$, and deduce that the even partial sums diverge to infinity.

7. $\sum_1^\infty 1/n^\alpha$ converges for $\alpha > 1$, diverges for $\alpha \leqslant 1$.

8. Nothing can be said if $l = 1$.

9. The power series diverges at $x = \frac{1}{4}$ by use of this comparison test with the series $\sum_1^\infty 1/n$, and converges at $x = -\frac{1}{4}$ by the alternating series test. (To show the sequence of absolute values is null, compare the ratio of consecutive terms with that for $n^{-1/3}$.)

16. If the nth partial sum of $\sum_1^\infty a_n$ is s_n and its sum to infinity is s, then the nth partial sum of $\sum_{N+1}^\infty a_n$ is $s_{N+n} - s_N$, which converges to $s - s_N$.

17. If $|x| < R$, choose r satisfying $|x| < r < R$, write $na_n x^{n-1} = n(x/r)^{n-1} a_n r^{n-1}$ and observe that $\sum_1^\infty n(x/r)^{n-1}$ is absolutely convergent by the ratio test, and $(a_n r^{n-1})_{n \geqslant 1}$ is a null sequence. If $|x| > R$, show $\sum_1^\infty na_n x^{n-1}$ is not absolutely convergent by comparison with $\sum_1^\infty a_n x^n$.

Chapter 4

4.17 (i) No limit. (ii) Limit exists $= 0$. (iii) No limit.

4.59 1. (i) Discontinuities at $x = n\pi$. (ii) Discontinuity at $x = 0$. (iii) Function is $|x|$ which is continuous everywhere. (iv) Discontinuities at $x = 1/n\pi$ and $x = 0$. (v) Continuous everywhere. (vi) Discontinuous everywhere. (vii) Continuous at $x = \frac{1}{2}$ only.

2. (i) 1. (ii) 1. (iii) $\frac{1}{2}$.

4. To prove the left-hand inequality, observe that the left-hand inequality of question 3 is true for all $x \geqslant 0$ (clearly from the definition of e^x), therefore the right-hand inequality of question 3 is true for all $x < 1$ (put $x = -u$ for $x < 0$); now put

$$x = \frac{u}{1+u} \qquad (u > -1)$$

and take logarithms. (Draw diagrams!)

6. The series converges conditionally. Use alternating series test. Show $(\log n)/n$ decreases for $n \geqslant 3$, e.g. by showing $(n+1)^n < n^{n+1}$ for $n \geqslant 3$ by 2.34.

7. Compare the Maclaurin series of the moduland with a geometric series.

Chapter 5

5.12
$$f'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x} \qquad (x \neq 0),$$
$$= 0 \qquad (x = 0).$$

Use 5.6 for $x \neq 0$ and 5.1 for $x = 0$. Theorem 5.3 says $f(x)$ is continuous, *not* $f'(x)$ continuous.

5.26 Maximum at $x = -2$, minimum at $x = 0$.

5.32 (i) 1. (ii) -1. (iii) $\frac{1}{2}$.

5.36 See remarks after 5.37.

5.39 (i) $e^x = e^2 e^{x-2}$. (ii) $\sin x = \cos(x - \frac{1}{2}\pi)$. (iii) $\sum_0^\infty (x+1)^n/2^{n+1}$.

5.44 1. (i) Discontinuous. (ii) Continuous, not differentiable. (iii) Differentiable.

2. (i) Maximum at $x = 0$. (ii) Minimum at $x = -1/\sqrt{2}$, maximum at $x = 1/\sqrt{2}$. (iii) Maxima at $x = \pm 1$, minimum at $x = 0$.

3. (ii) Differentiate

$$f(x) = \sin x - \frac{x(\pi - x)}{\pi}$$

twice. (iii) Observe that

$$\frac{\sin x}{x} \geqslant 1 - \frac{x^2}{6}$$

for all x, and

$$1 - \frac{x^2}{6} \geqslant \frac{\pi^2 - x^2}{\pi^2 + x^2}$$

if $x^2 \leqslant 12 - \pi^2$. If we put $x = \pi - u$ the inequality is equivalent to

$$\frac{\sin u}{u} \geqslant \frac{2\pi^2 - 3\pi u + u^2}{2\pi^2 - 2\pi u + u^2}$$

if $0 \leqslant u \leqslant \pi$. By the same method this inequality is true if

$$6\pi - 2\pi^2 u + 2\pi u^2 - u^3 \geqslant 0$$

which certainly holds for $0 \leqslant u \leqslant 2$. If we put $x = \pi + u$ the inequality is equivalent to

$$\frac{\sin u}{u} \leqslant \frac{2\pi^2 + 3\pi u + u^2}{2\pi^2 + 2\pi u + u^2}$$

if $u \geqslant 0$, and this holds for all $u \geqslant 0$ since the right-hand side is $\geqslant 1$.

4. (i) 2. (ii) $\frac{1}{2}$. (iii) $\log 2$. (iv) e^3.

5. Use L'Hôpital's rule (differentiating with respect to h). The second limit is $-f'''(x)$, on the assumption that $f(x)$ is thrice differentiable at x. The general case involves bionomial coefficients.

6. Consider

$$\frac{f'(x) - f'(c)}{x - c}$$

for x near c and use 5.24. $f''(c) < 0$ gives a local maximum at $x = c$. No conclusion can be drawn in case $f''(c) = 0$.

12. Equality is at $x = 0$ only ($\alpha \neq 0, 1$).

Chapter 6

6.2

$$L_D = \frac{N-1}{2N}, \; U_D = \frac{N+1}{2N}.$$

6.16 All three functions are integrable, (i) since monotonic, (ii) since continuous, (iii) since for any D including $1 \pm \varepsilon$, $2 \pm \varepsilon$ we have $U_D - L_D \leqslant 4\varepsilon$.

6.33 (i) -2. (ii) $e - 2$. (iii) $\frac{1}{2}(1 + e^\pi)$. (iv) $2\log 2 - 1$.

6.36 (i) $\pi/6$. (ii) $2 - 2\log 2$. (iii) 1. (iv) = (ii).

6.37 1. (i) Given $\varepsilon > 0$, choose D as follows. Take $x_1 = \varepsilon$. Choose

$x_2, \ldots, x_N$ to give a dissection D' of $[\varepsilon, 1]$ such that $U_{D'} - L_{D'} < \varepsilon$. ($\sin 1/x$ is integrable over $[\varepsilon, 1]$ since continuous on this interval.) We obtain $U_D - L_D < 3\varepsilon$ which shows $\sin 1/x$ is integrable over $[0, 1]$ by 6.13. (ii) This function is continuous therefore integrable.

4. If $|f(t)| \leq M$ for all $t \in [a, b]$, then

$$|F(x+h) - F(x)| \leq \int_x^{x+h} |f(t)|\, dt \leq hM$$

if $h > 0$.

7. Write $F(x) = \int_a^x f(t)\, dt$ and observe that

$$\int_a^b f(x)g(x)\, dx = F(b)g(b) - \int_a^b F(x)g'(x)\, dx.$$

Now use question 6. For the last part put $f(x) = \sin x$, $g(x) = 1/x$.

9. By continuity there exists $\delta > 0$ such that $f(x) \geq \frac{1}{2}f(c)$ for all $|c - x| \leq \delta$. Therefore

$$\int_a^b f(x)\, dx \geq \int_{c-\delta}^{c+\delta} \tfrac{1}{2}f(c)\, dx = \delta f(c) > 0.$$

See 6.11 (ii) for the last part.

10. (i) $\int_1^2 dx/x = \log 2$. (ii) $\int_1^3 dx/x = \log 3$. (iii) The logarithm of the sequence converges to $\int_1^2 \log x\, dx = 2\log 2 - 1$, therefore the sequence itself converges to $4/e$.

Chapter 7

7.3 1. (i) Converges only if $\alpha > 0$ when its value is $e^{-\alpha}/\alpha$. (ii) Diverges. (iii) Converges with value $\frac{1}{2}\pi$.

7.6 1. (i) Convergent with value -1. (ii) Divergent. (iii) Convergent with value π.

2. Use question 4 of 6.37.

7.12 1. (i) Convergent by comparison with $\int_1^\infty dx/x^3$ (see 7.2).

7.16 1. (i) $1/(1+x^2)$. (ii) $x/(1+x^2)$.

2. Use 6.25.

7.20 Make the substitution $t = x^2$ in the integral $\int_0^\infty t^{-\frac{1}{2}}e^{-t}\, dt$.

7.26 1. (i) Conditionally convergent for $0 < \alpha \leq 1$, absolutely convergent for $\alpha > 1$. Use the methods of 7.15, 7.18. (ii) Conditionally convergent. Substitute $x = \sqrt{t}$. (iii) Conditionally convergent. Substitute $x = \sqrt[3]{t}$. (iv) Convergent. Compare with $\int_0^1 x^{-\frac{1}{2}}\, dx$ at $x = 0$. (v) Conditionally convergent. Put $x = 1/t$. (vi) Divergent. (vii) Convergent. Compare with $\int_0^1 \log x\, dx$ at $x = 0$, $\int_0^\pi \log(\pi - x)\, dx$ at $x = \pi$. (viii) Divergent. The indefinite integral of $(\log x)/x$ is $\frac{1}{2}(\log x)^2$.

2. No in general. Consider question 1, (ii) and (iii). Yes if $f(x)$ is decreasing. Observe that

$$0 \leq f(X) \leq \int_{X-1}^{X} f(x)\, dx \to 0$$

as $X \to \infty$. No if $f(x) \geq 0$, consider e.g. $f(x) = 1 - 2^n |x - n|$ for x satisfying $|x - n| \leq 1/2^n$ and otherwise zero.

4. Use question 3.

9. (ii) Use question 8.

10. Use question 9 and the approximate value of γ given in 5.23.

11. Proceed as in 7.25.

12. (i) The logarithm of the sequence converges to $\int_0^1 \log x\, dx = -1$. Therefore the sequence itself converges to $1/e$. To justify this assertion observe that

$$\int_0^1 \log x\, dx < \frac{1}{n}\log\left(\frac{1}{n}\frac{2}{n}\cdots\frac{n}{n}\right) < \int_{1/n}^1 \log x\, dx$$

and use the sandwich principle. (ii) This sequence converges to $\int_0^1 dx/\sqrt{[x(1-x)]} = \pi$. To justify observe

$$\int_{1/(n+1)}^{n/(n+1)} \frac{dx}{\sqrt{[x(1-x)]}} < \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}} < \int_0^1 \frac{dx}{\sqrt{[x(1-x)]}}.$$

(Draw diagrams.)