# Appendix

## 1. Euler's proof that $\sum_1^\infty 1/n^2 = \pi^2/6$

The proof depends on the fact that, if the roots of the polynomial equation

$$a_0 + a_1 x + \cdots + a_n x^n = 0$$

are $\alpha_1, \alpha_2, \ldots, \alpha_n$, then

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \cdots + \frac{1}{\alpha_n} = -\frac{a_1}{a_0}.$$

Now the roots of the equation

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots = 0$$

are $\pm\pi, \pm 2\pi, \ldots, \pm n\pi, \ldots$, and so the roots of

$$1 - \frac{x}{3!} + \frac{x^2}{5!} - \cdots = 0$$

are $\pi^2, 4\pi^2, \ldots, n^2\pi^2, \ldots$. Hence

$$\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \cdots + \frac{1}{n^2\pi^2} + \cdots = \frac{1}{3!},$$

which gives

$$\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}. \qquad \square$$

## 2. Proof that $\sum_1^\infty \sin nx/n = \frac{1}{2}(\pi - x)$ for all $0 < x < 2\pi$

We proved in Chapter 5 that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$



for all $-1 < x \leq 1$ (see 5.22 and 5.42). If we put $x = -e^{i\theta}$, where $0 < \theta < 2\pi$, we get

$$\log(1 - e^{i\theta}) = -e^{i\theta} - \frac{e^{2i\theta}}{2} - \frac{e^{3i\theta}}{3} - \frac{e^{4i\theta}}{4} - \cdots.$$

Taking imaginary parts gives

$$\sum_1^\infty \frac{\sin n\theta}{n} = -\arg(1 - e^{i\theta}) = \frac{\pi - \theta}{2},$$

since, e.g.

$$\begin{aligned} 1 - e^{i\theta} &= -e^{\frac{1}{2}i\theta}(e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}i\theta}) \\ &= -e^{\frac{1}{2}i\theta}(2i \sin\tfrac{1}{2}\theta) \\ &= 2\sin\tfrac{1}{2}\theta\, e^{\frac{1}{2}i(\theta - \pi)}. \end{aligned} \qquad \square$$

Observe that taking real parts gives

$$\sum_1^\infty \frac{\cos n\theta}{n} = -\log(2\sin\tfrac{1}{2}\theta)$$

for all $0 < \theta < 2\pi$.