7 Singular integrals

Up to now we have considered only integrals of bounded functions over bounded intervals. We now wish to allow the possibility of integrating unbounded functions over unbounded intervals. Many of the more interesting integrals of analysis come into this category: for example, the Laplace transform $F(x)$ of $f(t)$ is defined as

$$F(x) = \int_0^\infty e^{-xt} f(t)\, dt,$$

and the continuous factorial function $x!$ is defined as

$$x! = \int_0^\infty t^x e^{-t}\, dt.$$

We shall call integrals such as these *singular* integrals and classify them into three kinds. Singular integrals of the *first* kind are those which involve integration of a bounded function over an unbounded interval, e.g. $\int_1^\infty dx/x$. Singular integrals of the *second* kind will involve integration of an unbounded function over a bounded interval, e.g. $\int_0^1 dx/x$. Integrals of the *third* kind combine the characteristics of the other two kinds, e.g. $\int_0^\infty dx/x$.

We shall refer to points near which an integrand is unbounded as *singularities*, e.g. $\int_0^1 dx/x$ has a singularity at $x = 0$. Integrals such as $\int_1^2 dx/x$, where there are no singularities and the range of integration is bounded will be called *ordinary* integrals.

Singular integrals will be defined by regarding them as limiting cases of ordinary integrals. We shall have to allow for the possibility of non-existence of limits by introducing the notions of convergence and divergence.

7.1 Definition

Suppose that $f(x)$ is continuous for all $x \geqslant a$, where a is fixed. We say $\int_a^\infty f(x)\, dx$ *converges*, or is *convergent*, if

$$\lim_{X\to\infty} \int_a^X f(x)\, dx$$

exists and is finite. When $\int_a^\infty f(x)\, dx$ converges we define its value, also denoted by $\int_a^\infty f(x)\, dx$, to be

$$\int_a^\infty f(x)\, dx = \lim_{X\to\infty} \int_a^X f(x)\, dx.$$

If $\int_a^\infty f(x)\, dx$ fails to converge, we shall say it *diverges*, or is *divergent*. □

7.2 Example

$$\int_1^\infty dx/x^\alpha \qquad (\alpha \text{ fixed}).$$

We have

$$\int_1^X \frac{dx}{x^\alpha} = \left.\frac{x^{-\alpha+1}}{-\alpha+1}\right|_1^X = \frac{X^{-\alpha+1}-1}{-\alpha+1}$$

$$\to \begin{cases} \dfrac{1}{\alpha-1} & \text{as } X\to\infty \text{ if } \alpha>1, \\ \infty & \text{as } X\to\infty \text{ if } \alpha<1. \end{cases}$$

If $\alpha = 1$,

$$\int_1^X \frac{dx}{x} = \log x \Big|_1^X = \log X \to \infty \qquad \text{as } X\to\infty.$$

Hence, $\int_1^\infty dx/x^\alpha$ converges if $\alpha > 1$, and diverges if $\alpha \leqslant 1$. For $\alpha > 1$ we have

$$\int_1^\infty \frac{dx}{x^\alpha} = \frac{1}{\alpha-1}.$$ □

7.3 Exercises

1. Which of the following integrals converge? Find the values of those which converge.

(i) $\int_1^\infty e^{-\alpha x}\, dx$ (ii) $\int_0^\infty \frac{dx}{1+x}$ (iii) $\int_0^\infty \frac{dx}{1+x^2}$

2. Show that if $f(x)$, $g(x)$ are continuous for $x \geqslant a$, and $\int_a^\infty f(x)\,dx$, $\int_a^\infty g(x)\,dx$ both converge, then also $\int_a^\infty (\alpha f(x)+\beta g(x))\,dx$ converges for any constants α, β and its value is given by

$$\int_a^\infty (\alpha f(x)+\beta g(x))\,dx = \alpha\int_a^\infty f(x)\,dx + \beta\int_a^\infty g(x)\,dx.$$

3. Show that, if $f(x)$ is continuous for $x \geqslant a$, and if $b > a$, then $\int_a^\infty f(x)\,dx$ converges if and only if $\int_b^\infty f(x)\,dx$ converges. □

7.4 Definition

Suppose that $f(x)$ is continuous for $a < x \leqslant b$, where $a < b$ are fixed. We say $\int_a^b f(x)\,dx$ *converges*, or is *convergent*, if

$$\lim_{\varepsilon\to 0_+}\int_{a+\varepsilon}^b f(x)\,dx$$

exists finite. When $\int_a^b f(x)\,dx$ converges, we define its *value*, also denoted by $\int_a^b f(x)\,dx$, to be

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0_+}\int_{a+\varepsilon}^b f(x)\,dx.$$

If $\int_a^b f(x)\,dx$ fails to converge, we shall say it *diverges*, or is *divergent*. □

7.5 Example

$\int_0^1 dx/x^\alpha$ (α fixed).

We have

$$\int_\varepsilon^1 \frac{dx}{x^\alpha} = \left.\frac{x^{-\alpha+1}}{-\alpha+1}\right|_\varepsilon^1 = \frac{1-\varepsilon^{-\alpha+1}}{-\alpha+1} \to \begin{cases}\dfrac{1}{1-\alpha} & \text{as } \varepsilon\to 0_+ \text{ if } \alpha<1,\\ \infty & \text{as } \varepsilon\to 0_+ \text{ if } \alpha>1.\end{cases}$$

If $\alpha = 1$,

$$\int_\varepsilon^1 \frac{dx}{x} = \log x\Big|_\varepsilon^1 = -\log\varepsilon \to \infty \quad \text{as } \varepsilon\to 0_+.$$

Hence, $\int_0^1 dx/x^\alpha$ converges if $\alpha < 1$ and diverges if $\alpha \geqslant 1$. For $\alpha < 1$ its value is

$$\int_0^1 \frac{dx}{x^\alpha} = \frac{1}{1-\alpha}.$$

Observe that, for $\alpha \leqslant 0$, $\int_0^1 dx/x^\alpha$ is an ordinary integral. □

7.6 Exercises

1. Say which integrals converge and find their values.

(i) $\int_0^1 \log x\,dx$ (Integrate by parts)

(ii) $\int_0^{\frac{1}{2}\pi} \tan x\,dx$

(iii) $\int_{-1}^1 \frac{dx}{\sqrt{(1-x^2)}}$ (Substitute $x = \sin t$).

2. Show that if $f(x)$ is continuous over $a \leqslant x \leqslant b$, then

$$\lim_{\varepsilon\to 0_+}\int_{a+\varepsilon}^b f(x)\,dx = \int_a^b f(x)\,dx$$

where $\int_a^b f(x)\,dx$ is the ordinary integral. □

For integrals of the third kind we split the range of integration up to produce a sum of integrals of the first and second kind. We say they converge when *all* the component integrals converge, and the value is the sum of the values of the components.

7.7 Example

$\int_0^\infty dx/x^\alpha$.

Write

$$\int_0^\infty \frac{dx}{x^\alpha} = \int_0^1 \frac{dx}{x^\alpha} + \int_1^\infty \frac{dx}{x^\alpha}.$$

The first integral diverges for all $\alpha \geqslant 1$ (7.5), and the second diverges for all $\alpha \leqslant 1$ (7.2), so $\int_0^\infty dx/x$ diverges for all α. □

We will give examples of convergent integrals of the third kind when we have developed a few tests for convergence.

The theory of convergence of singular integrals has many parallels with the theory of infinite series. We shall carry over much of the terminology and develop the theory in analogous fashion. For example, we shall begin by considering integrals with positive integrand. These are the analogues of series of positive terms. Later we define a notion of absolute convergence for integrals with general integrand. Also, we shall show there is a strong connection between integrals $\int_1^\infty f(x)\,dx$ of the first kind, and the corresponding series $\sum_1^\infty f(n)$.

7.8 Theorem

If $f(x) \geqslant 0$ and is continuous for all $x \geqslant a$, then $\int_a^\infty f(x)\,dx$ converges if and only if $\int_a^X f(x)\,dx$ is bounded for $X \geqslant a$.

Proof If we write $F(X) = \int_a^X f(x)\,dx$, then $F(X)$ is an increasing function of X, so converges to a finite limit or diverges to infinity according as it is bounded or not. □

The corresponding theorem for integrals of the second kind is that, if $f(x) \geqslant 0$ is continuous for $a < x \leqslant b$, then $\int_a^b f(x)\,dx$ converges if and only if $\int_{a+\varepsilon}^b f(x)\,dx$ is bounded for $\varepsilon > 0$ ($\varepsilon \leqslant b - a$).

7.9 Comparison test

If $0 \leqslant f(x) \leqslant g(x)$ are continuous for $x \geqslant a$, then convergence of $\int_a^\infty g(x)\,dx$ implies convergence of $\int_a^\infty f(x)\,dx$.

Proof If $\int_a^\infty g(x)\,dx$ converges, then, by 7.8, $\int_a^X g(x)\,dx$ is bounded for $X \geqslant a$, but therefore $\int_a^X f(x)\,dx$ is bounded for $X \geqslant a$, since

$$\int_a^X f(x)\,dx \leqslant \int_a^X g(x)\,dx,$$

and hence, by 7.8 again, $\int_a^\infty f(x)\,dx$ converges. □

7.10 Corollary

Under the same hypotheses, the divergence of $\int_a^\infty f(x)\,dx$ implies the divergence of $\int_a^\infty g(x)\,dx$. □

7.11 Examples

1. $\int_1^\infty dx/(1+x)$
For all $x \geqslant 1$ we have

$$\frac{1}{1+x} \geqslant \frac{1}{2x}$$

and $\int_1^\infty dx/2x$ is divergent (see 7.2, 7.3). Hence $\int_1^\infty dx/(1+x)$ is divergent by the comparison test (7.10).
2. $\int_1^\infty dx/(1+x^2)$.
For all $x \geqslant 1$ we have

$$\frac{1}{1+x^2} \leqslant \frac{1}{x^2}$$

and $\int_1^\infty dx/x^2$ is convergent (see 7.2). Hence $\int_1^\infty dx/(1+x^2)$ is convergent by the comparison test (7.9).
Both the above integrals can of course be treated by going back to the definition (see 7.3). □

7.12 Exercises

1. Discuss the convergence of the following integrals.

(i) $$\int_1^\infty \frac{dx}{1+x^3}$$

(ii) $$\int_1^\infty e^{-x^2}\,dx \qquad (e^{-x^2} \leqslant e^{-x}).$$

2. Show that, if $0 \leqslant f(x) \leqslant g(x)$ are continuous for $x \geqslant a$, then

$$\int_a^\infty f(x)\,dx \leqslant \int_a^\infty g(x)\,dx,$$

with the obvious interpretation if either integral diverges.
3. Formulate a comparison test for integrals of the second kind, and use it to discuss the convergence of the following integrals.

(i) $$\int_0^{\frac{1}{2}\pi} \frac{dx}{\sin x} \qquad (\sin x \leqslant x)$$

(ii) $$\int_0^1 \frac{dx}{\log(1+x)} \qquad (\log(1+x) \leqslant x)$$ □

7.13 Definition

Suppose $f(x)$ is continuous for $x \geqslant a$. We say the integral $\int_a^\infty f(x)\,dx$ is *absolutely convergent*, or *converges absolutely*, if $\int_a^\infty |f(x)|\,dx$ converges. □

7.14 Theorem

Absolute convergence implies convergence.

Proof Let

$$f^+(x) = \max\{f(x), 0\} = \tfrac{1}{2}(|f(x)| + f(x)),$$
$$f^-(x) = \max\{-f(x), 0\} = \tfrac{1}{2}(|f(x)| - f(x)).$$

Then $f^+(x)$, $f^-(x)$ are continuous and

$$f^+(x) \geqslant 0, \quad f^-(x) \geqslant 0,$$
$$f^+(x) - f^-(x) = f(x),$$
$$f^+(x) + f^-(x) = |f(x)|.$$

Therefore $f^+(x) \leqslant |f(x)|$, $f^-(x) \leqslant |f(x)|$ and so, by the comparison test, $\int_a^\infty f^+(x)\,dx$, $\int_a^\infty f^-(x)\,dx$ both converge. Hence $\int_a^\infty f(x)\,dx$ converges by 7.3. □

7.15 Examples

$$\int_1^\infty \frac{\sin x}{x^2}\,dx, \qquad \int_1^\infty \frac{\cos x}{x^2}\,dx.$$

We have

$$\left|\frac{\sin x}{x^2}\right| \leqslant \frac{1}{x^2}$$

for all $x \geqslant 1$, so $\int_1^\infty (\sin x)/x^2\,dx$ is absolutely convergent by the comparison test, and therefore convergent. Similarly for $\int_1^\infty (\cos x)/x^2\,dx$. □

7.16 Exercises

1. Show that the following integrals are absolutely convergent and find their values.

$$\text{(i)} \int_0^\infty e^{-xt}\sin t\,dt \qquad \text{(ii)} \int_0^\infty e^{-xt}\cos t\,dt \qquad (x > 0)$$

2. Show that, if $\int_a^\infty f(x)\,dx$ is absolutely convergent, then

$$\left|\int_a^\infty f(x)\,dx\right| \leqslant \int_a^\infty |f(x)|\,dx.$$

3. Formulate a definition of absolute convergence for integrals of the second kind, and prove absolute convergence implies convergence. □

7.17 Definition

$\int_a^\infty f(x)\,dx$ is *conditionally convergent* if convergent but not absolutely. □

7.18 Example

$$\int_1^\infty \frac{\sin x}{x}\,dx \qquad \text{is conditionally convergent.}$$

Proof Integrating by parts we have

$$\int_1^X \frac{\sin x}{x}\,dx = -\frac{\cos x}{x}\bigg|_1^X - \int_1^X \frac{\cos x}{x^2}\,dx$$
$$= \cos 1 - \frac{\cos X}{X} - \int_1^X \frac{\cos x}{x^2}\,dx$$
$$\to \cos 1 - \int_1^\infty \frac{\cos x}{x^2}\,dx$$

as $X \to \infty$. (See 7.15.) Hence $\int_1^\infty (\sin x)/x\,dx$ is convergent.
However,

$$\left|\frac{\sin x}{x}\right| \geqslant \frac{\sin^2 x}{x} = \frac{1 - \cos 2x}{2x},$$

and

$$\int_1^X \frac{1 - \cos 2x}{2x}\,dx = \frac{1}{2}\int_1^X \frac{dx}{x} - \frac{1}{2}\int_1^X \frac{\cos 2x}{x}\,dx$$
$$\to \infty$$

as $X \to \infty$, since $\int_1^\infty (1/x)\,dx$ is divergent, and $\int_1^\infty (\cos 2x)/x\,dx$ is convergent (by the same argument as that used above to prove $\int_1^\infty (\sin x)/x \, dx$ is convergent). Hence, by the comparison test, $\int_1^\infty (\sin x)/x \, dx$ is not absolutely convergent.

7.19 The gamma function

We shall now illustrate the techniques of testing integrals of the third kind for convergence by giving a treatment of the integral $\int_0^\infty t^x e^{-t} \, dt$ for $x > -1$.

If $x \geqslant 0$, we have an integral of the first kind.

If $x < 0$, we have an integral of the third kind with a singularity at $t = 0$.

Consider $\int_1^\infty t^x e^{-t} \, dt$.

We have

$$t^x e^{-t} = (t^x e^{-\frac{1}{2}t}) e^{-\frac{1}{2}t}$$

and $t^x e^{-\frac{1}{2}t} \to 0$ as $t \to \infty$; therefore there exists M such that

$$t^x e^{-\frac{1}{2}t} \leqslant M$$

for all $t \geqslant 1$. It follows that

$$t^x e^{-t} \leqslant M e^{-\frac{1}{2}t}$$

for all $t \geqslant 1$. Hence $\int_1^\infty t^x e^{-t} \, dt$ converges for all x by comparison with $\int_1^\infty e^{-\frac{1}{2}t} \, dt$ (see 7.3).

Now consider $\int_0^1 t^x e^{-t} \, dt$.

We have

$$t^x e^{-t} \leqslant t^x$$

for all $0 \leqslant t \leqslant 1$, and $\int_0^1 t^x \, dt$ converges for $x > -1$ (see 7.5). Hence $\int_0^1 t^x e^{-t} \, dt$ converges for all $x > -1$ by the comparison test.

Observe that, for any integer $n \geqslant 0$,

$$\begin{aligned} I_n &= \int_0^\infty t^n e^{-t} \, dt \\ &= -t^n e^{-t} \Big|_0^\infty + n \int_0^\infty t^{n-1} e^{-t} \, dt \\ &= n I_{n-1} \\ &\ \ \cdots \\ &= n! \, I_0 \\ &= n! \end{aligned}$$

So it is natural to define

$$x! = \int_0^\infty t^x e^{-t} \, dt$$

for all real $x > -1$. Alternatively one defines the *gamma function*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt$$

for $x > 0$, so that $\Gamma(n) = (n-1)!$ for all integers $n \geqslant 1$.

7.20 Exercise

Show $(-\frac{1}{2})! = 2\int_0^\infty e^{-t^2} \, dt (= \sqrt{\pi})$. □

7.21 Integral test (for series)

If $f(x) \geqslant 0$ is continuous and decreasing for $x \geqslant 1$, then convergence of the integral $\int_1^\infty f(x) \, dx$ is necessary and sufficient for convergence of the series $\sum_1^\infty f(n)$.

Proof We have

$$f(n+1) \leqslant f(x) \leqslant f(n)$$

for all $n \leqslant x \leqslant n+1$. Therefore

$$\int_n^{n+1} f(n+1) \, dx \leqslant \int_n^{n+1} f(x) \, dx \leqslant \int_n^{n+1} f(n) \, dx,$$

i.e.

$$f(n+1) \leqslant \int_n^{n+1} f(x) \, dx \leqslant f(n).$$

Therefore, if we sum over $n = 1, 2, \ldots, N$, we obtain

$$\sum_2^{N+1} f(n) \leqslant \int_1^{N+1} f(x) \, dx \leqslant \sum_1^N f(n).$$

It follows that $\sum_1^N f(n)$ is bounded over N if and only if $\int_1^X f(x) \, dx$ is bounded over X. Hence the series $\sum_1^\infty f(n)$ converges if and only if the integral $\int_1^\infty f(x) \, dx$ does. □

7.22 Example

$\sum_1^\infty (1/n^\alpha)$.

The function $f(x) = 1/x^\alpha$ satisfies the conditions of 7.21 if $\alpha \geqslant 0$. Therefore, by 7.2, we find $\sum_1^\infty (1/n^\alpha)$ is convergent if $\alpha > 1$, and divergent if $0 \leqslant \alpha \leqslant 1$. In fact, $\sum_1^\infty (1/n^\alpha)$ diverges for all $\alpha \leqslant 1$, since, for $\alpha < 0$, the nth term $1/n^\alpha \not\to 0$. □

7.23 Euler's limit

The sequence (γ_n), where

$$\gamma_n = 1 + \tfrac{1}{2} + \cdots + \frac{1}{n} - \log n$$

is decreasing and satisfies $0 \leq \gamma_n \leq 1$ for all n, hence is convergent to a finite limit γ, also satisfying $0 \leq \gamma \leq 1$, as $n \to \infty$. γ is known as Euler's constant and its value is 0.577 to 3 decimal places.

To justify these assertions, we argue as in the proof of 7.21 for the particular case $f(x) = 1/x$. We arrive at the inequalities

$$\sum_2^{N+1} \frac{1}{n} \leq \int_1^{N+1} \frac{dx}{x} \leq \sum_1^N \frac{1}{n},$$

which give

$$(\log N \leq) \log (N+1) \leq \sum_1^N \frac{1}{n} \leq 1 + \log N$$

and therefore $0 \leq \gamma_N \leq 1$. To show γ_n decreases, we observe that

$$\begin{aligned} \gamma_n - \gamma_{n+1} &= -\frac{1}{n+1} - \log n + \log (n+1) \\ &= \int_n^{n+1} \left(\frac{1}{x} - \frac{1}{n+1}\right) dx \\ &\geq 0, \end{aligned}$$

since the integrand is positive. □

7.24 Estimates of $\sum_1^N (1/n)$

The above analysis can be used to give us some idea of how large the partial sums of the harmonic series $\sum_1^\infty (1/n)$ are. We know this series diverges to infinity. We shall see, however, that it does so remarkably slowly.

In fact,

$$\begin{aligned} \sum_1^N \frac{1}{n} &= \gamma_N + \log N \\ &\leq 1 + \log N. \end{aligned}$$

Now $\log 10 = 2.3026$ to 4 decimal places, so e.g. $\sum_1^{100} (1/n) < 6$, $\sum_1^{10^6} (1/n) < 15$. On the other hand, we also have

$$\sum_1^N \frac{1}{n} \geq \gamma + \log N,$$

which shows that, if e.g. we want $\sum_1^N (1/n) > 100$, we require N to be in the region of $e^{100-\gamma} = 1.51 \times 10^{43}$ to 3 significant figures.

7.25 Rearrangements of $\sum_1^\infty (-1)^{n-1}/n$

We can use Euler's limit to find the sum of the alternating harmonic series $\sum_1^\infty (-1)^{n-1}/n$, and various rearrangements of it, as follows.

$$\begin{aligned} \sum_1^{2N} (-1)^{n-1}/n &= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots - \frac{1}{2N} \\ &= \sum_1^{2N} (1/n) - \sum_1^N (1/n) \\ &= (\log 2N + \gamma_{2N}) - (\log N + \gamma_N) \\ &= \log 2 + \gamma_{2N} - \gamma_N \\ &\to \log 2 \end{aligned}$$

as $N \to \infty$. Hence $\sum_1^\infty (-1)^{n-1}/n = \log 2$.

Consider e.g. the rearrangement

$$1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \cdots$$

The $3N$th partial sum is

$$\begin{aligned} 1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \cdots + \frac{1}{4N-3} + \frac{1}{4N-1} - \frac{1}{2N} \\ = \sum_1^{4N} (1/n) - \tfrac{1}{2}\sum_1^{2N} (1/n) - \tfrac{1}{2}\sum_1^N (1/n) \\ = (\log 4N + \gamma_{4N}) - \tfrac{1}{2}(\log 2N + \gamma_{2N}) - \tfrac{1}{2}(\log N + \gamma_N) \\ = \tfrac{3}{2} \log 2 + \gamma_{4N} - \tfrac{1}{2}\gamma_{2N} - \tfrac{1}{2}\gamma_N \\ \to \tfrac{3}{2} \log 2 \end{aligned}$$

as $N \to \infty$. Hence

$$1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \cdots = \tfrac{3}{2} \log 2.$$

□

7.26 Miscellaneous exercises

1. Discuss the convergence of the following integrals.

(i) $\displaystyle\int_1^\infty \frac{\sin x}{x^\alpha} dx \quad (\alpha > 0)$ (ii) $\displaystyle\int_1^\infty \sin (x^2)\, dx$

(iii) $\int_1^\infty x \sin(x^3)\,dx$ (iv) $\int_0^\infty \frac{\sin x}{x^{3/2}}\,dx$

(v) $\int_0^1 \frac{1}{x}\sin\left(\frac{1}{x}\right) dx$ (vi) $\int_0^{\frac{1}{2}\pi} \sec x\,dx$

(vii) $\int_0^\pi \log \sin x\,dx$ (viii) $\int_0^1 \frac{\log x}{x}\,dx$

2. Does $\int_1^\infty f(x)\,dx$ convergent imply $f(x) \to 0$ as $x \to \infty$? In general? If $f(x)$ is decreasing? If $f(x) \geqslant 0$?

3. Show that

$$\int_0^{\frac{1}{2}\pi} \log \sin x\,dx = \int_0^{\frac{1}{2}\pi} \log \cos x\,dx \qquad (u = \tfrac{1}{2}\pi - x)$$
$$= \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \log(\tfrac{1}{2}\sin 2x)\,dx$$
$$= -\tfrac{1}{2}\pi \log 2.$$

4. Show that

$$\int_0^{2\pi} \log|1 - e^{i\theta}|\,d\theta = 0.$$

5. Show that

$$\int_0^\infty \frac{dx}{1+x^3} = \int_0^\infty \frac{x\,dx}{1+x^3} \qquad (x = 1/u)$$
$$= \tfrac{1}{2}\int_0^\infty \frac{dx}{1 - x + x^2}$$
$$= 2\pi/3\sqrt{3}.$$

6. Show

(i) $$\int_0^\infty \frac{dx}{1+x^4} = \int_0^\infty \frac{x^2\,dx}{1+x^4} \qquad (x = 1/u)$$

(ii) $$\int_0^\infty \frac{x\,dx}{1+x^4} = \tfrac{1}{2}\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{4} \qquad (u = x^2)$$

(iii) $$\int_0^\infty \frac{dx}{1+x^4} = \tfrac{1}{2}\int_0^\infty \frac{1+x^2}{1+x^4}\,dx$$
$$= \tfrac{1}{2}\int_0^\infty \frac{dx}{1+\sqrt{2}x+x^4} + \frac{1}{\sqrt{2}}\int_0^\infty \frac{x\,dx}{1+x^4}$$
$$= \pi/2\sqrt{2}.$$

7. Consider the series $\sum_2^\infty 1/n^\alpha(\log n)^\beta$.

(i) Show that, if $\alpha > 1$, it converges for all β. *Hint* Let $\alpha = 1 + \varepsilon$ $(\varepsilon > 0)$ and write

$$\frac{1}{n^\alpha(\log n)^\beta} = \frac{1}{n^{1+\frac{1}{2}\varepsilon}}\,\frac{1}{n^{\frac{1}{2}\varepsilon}(\log n)^\beta}$$

and observe that $1/n^{\frac{1}{2}\varepsilon}(\log n)^\beta \to 0$ and $\sum_1^\infty 1/n^{1+\frac{1}{2}\varepsilon}$ converges.

(ii) Show that, if $\alpha < 1$, it diverges for all β. *Hint* Let $\alpha = 1 - \varepsilon$ $(\varepsilon > 0)$ and argue similarly.

(iii) Show that, if $\alpha = 1$, it converges for all $\beta > 1$ and diverges for all $\beta \leqslant 1$. *Hint* Use the integral test (7.21).

8. Show that $\sum_{N+1}^\infty (1/n^2) \leqslant 1/N$. *Hint* Observe that

$$\int_{n-1}^n \frac{dx}{n^2} \leqslant \int_{n-1}^n \frac{dx}{x^2}.$$

9. Let $\gamma_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$. Show that

$$\gamma_n - \gamma_{n+1} = \frac{1}{n+1}\int_0^1 \frac{1-t}{n+t}\,dt.$$

Deduce that

(i) $$\gamma_n - \gamma_{n+1} \leqslant \frac{1}{2n^2},$$

(ii) $$\gamma_n - \gamma \leqslant \frac{1}{2(n-1)},$$

where $\gamma = \lim_{n\to\infty} \gamma_n$.

10. Show that $\sum_1^{1000} (1/n) = 6.908$ to 3 decimal places. (Assume $\log 10 = 2.3026$ to 4 decimal places.)

11. Use Euler's limit to show that

(i) $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \cdots = \frac{1}{2}\log 2$.

(ii) $1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$ diverges to infinity.

12. By considering Riemann sums for suitable singular integrals, find the limits of the following sequences.

(i) $$\frac{(n!)^{1/n}}{n}$$

(ii) $$\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{[2(n-1)]}} + \frac{1}{\sqrt{[3(n-2)]}} + \cdots + \frac{1}{\sqrt{n}}$$

(cf. 3.38, question 12).