14. Prove the following assertions.

(i) $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \dfrac{x+y}{1-xy}$.

(ii) $2 \tan^{-1} \frac{1}{5} = \tan^{-1} \frac{5}{12}$.

(iii) $4 \tan^{-1} \frac{1}{5} = \tan^{-1} \frac{120}{119}$.

(iv) $\tan^{-1} \frac{120}{119} - \tan^{-1} 1 = \tan^{-1} \frac{1}{239}$.

(v) $\frac{1}{4}\pi = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239}$.

(v) is known as Machin's formula, and has been used to calculate π to vast numbers of decimal places by using the expansion

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

(See 5.23.)

15. Prove that, for any integer $N \geqslant 1$,

$$\mathrm{e} = \sum_0^{N-1} \frac{1}{n!} + \frac{\mathrm{e}^{u_N}}{N!}$$

for some $0 < u_N < 1$. Deduce that e is irrational. *Hint:* If $\mathrm{e} = p/q$ where p, q are positive integers, then e^{u_N}/N is an integer for all $N > q$.

6 Integral calculus

There are two essentially different ways of looking at integration. One can either regard it as the inverse process of differentiation, or one can regard it as a kind of continuous summation.

The first view gives rise to the 'indefinite integral' or 'primitive' of a function $f(x)$ defined as any function $F(x)$ whose derivative $F'(x) = f(x)$ (see 5.19). The second view gives rise to the 'definite integral' of a function $f(x)$ which is defined as the limiting value of sums of the form $\sum f(x)\, \mathrm{d}x$ as $\mathrm{d}x \to 0$. The definite integral $\int_a^b f(x)\, \mathrm{d}x$ admits a geometrical interpretation as the area under the graph of $y = f(x)$ between the limits a and b (see Fig. 6.1).

The two kinds of integral are of course intimately related. The definite integral is expressible in terms of the indefinite integral as

$$\int_a^b f(x)\, \mathrm{d}x = F(b) - F(a),$$

where $F(x)$ is any indefinite integral of $f(x)$. On the other hand, the definite integral of $f(x)$ can be used to define an indefinite integral $F(x)$ of $f(x)$ by writing

$$F(x) = \int_a^x f(t)\, \mathrm{d}t.$$

These two results are collectively known as the fundamental theorem of calculus, and it is the main object of this chapter to give a rigorous proof this theorem for continuous functions.

The difficult problem is to give a rigorous definition of $\lim_{\mathrm{d}x \to 0} \sum f(x)\, \mathrm{d}x$ since this is really rather a sophisticated kind of limit. We shall need a more adequate notation for $\sum f(x)\, \mathrm{d}x$ which gives a more detailed description of how the interval $[a, b]$ is to be subdivided into small pieces $\mathrm{d}x$. We shall then need to find a precise way of describing how $\sum f(x)\, \mathrm{d}x$ tends to a limit as the small pieces $\mathrm{d}x$ all tend to zero simultaneously. The discussion will be motivated by thinking of the definite integral as the area under the graph.

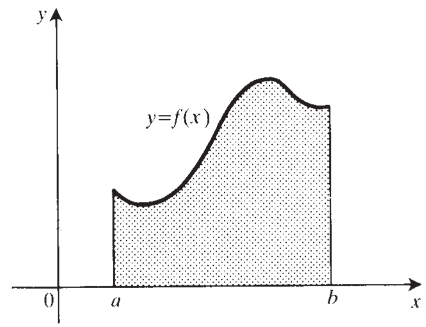


Fig. 6.1

6.1 Definitions

A *dissection* D of the interval $[a, b]$ is any finite set of points $x_0, x_1, x_2, \ldots, x_N$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_N = b.$$

Given any bounded function $f(x)$ over the interval $[a, b]$, we define its *upper sum* and *lower sum* for the dissection D, denoted by U_D and L_D, to be

$$U_D = \sum_1^N M_n(x_n - x_{n-1}),$$

$$L_D = \sum_1^N m_n(x_n - x_{n-1}) \ .$$

where

$$M_n = \sup_{x_{n-1} \leq x \leq x_n} f(x),$$

$$m_n = \inf_{x_{n-1} \leq x \leq x_n} f(x)$$

(see Fig. 6.2). □

Observe that U_D, L_D approximate the area under the curve from above and below by rectilinear areas.

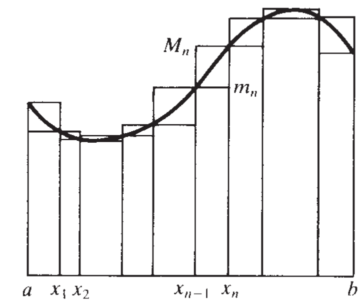


Fig. 6.2

6.2 Exercise

Compute U_D, L_D for $f(x) = x$ over the interval $[0, 1]$ corresponding to the dissection D given by

$$0 < \frac{1}{N} < \frac{2}{N} < \cdots < \frac{N-1}{N} < 1.$$ □

6.3 Lemma

$$m(b-a) \leq L_D \leq U_D \leq M(b-a)$$

where

$$M = \sup_{a \leq x \leq b} f(x),$$

$$m = \inf_{a \leq x \leq b} f(x).$$ □

Proof Clearly

$$m \leq m_n \leq M_n \leq M$$

for all n, and therefore

$$m \sum_1^N (x_n - x_{n-1}) \leq \sum_1^N m_n(x_n - x_{n-1}) \leq \sum_1^N M_n(x_n - x_{n-1})$$

$$\leq M \sum_1^N (x_n - x_{n-1}),$$

which is the result. □

6.4 Definition

We say D' is a *refinement* of D, and we write $D' > D$, if D' can be obtained from D by adding further points. □

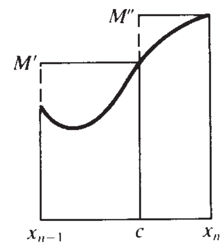


Fig. 6.3

For example, if D_N is the dissection of $[a, b]$ into N equal subintervals, then

$$D_1 < D_2 < D_4 < \cdots < D_{2^N} < \cdots.$$

6.5 Lemma

If $D' > D$, then $U_{D'} \leqslant U_D$ and $L_{D'} \geqslant L_D$. □

Proof Suppose D is

$$a = x_0 < x_1 < x_2 < \cdots < x_N = b$$

and that D' adds one further point c between x_{n-1} and x_n. Let

$$M' = \sup_{x_{n-1} \leqslant x \leqslant c} f(x),$$

$$M'' = \sup_{c \leqslant x \leqslant x_n} f(x)$$

(see Fig. 6.3). Then clearly $M' \leqslant M_n$, $M'' \leqslant M_n$, and therefore

$$\begin{aligned} U_{D'} &= \sum_{r \neq n} M_r(x_r - x_{r-1}) + M'(c - x_{n-1}) + M''(x_n - c) \\ &\leqslant \sum_{r \neq n} M_r(x_r - x_{r-1}) + M_n(c - x_{n-1}) + M_n(x_n - c) \\ &= U_D. \end{aligned}$$

Hence clearly $U_{D'} \leqslant U_D$ for any number of added points.

The proof of $L_{D'} \geqslant L_D$ is similar. □

6.6 Definition

Given any two dissections D, D' we define their *common refinement*, denoted by $D \cup D'$, to be the dissection obtained by combining all the points of D with all the points of D'. □

6.7 Lemma

For any D, D' we have $L_D \leqslant U_{D'}$. □

Proof Lemmas 6.3 and 6.5 give

$$L_D \leqslant L_{D \cup D'} \leqslant U_{D \cup D'} \leqslant U_{D'}.$$ □

6.8 Definition

The *upper and lower integrals* of bounded $f(x)$ over $[a, b]$, denoted by U, L, are defined to be

$$U = \inf_D U_D,$$

$$L = \sup_D L_D.$$ □

Observe that U is the limiting value of rectilinear areas exterior to the area under the graph, L is the limiting value of interior rectilinear areas.

6.9 Lemma

$$m(b-a) \leqslant L \leqslant U \leqslant M(b-a).$$

Proof follows from 6.3 and 6.7. □

6.10 Definition

We say bounded $f(x)$ is *integrable* over $[a, b]$ if $L = U$ and when this occurs we define the *definite integral* of $f(x)$ over $[a, b]$, denoted by $\int_a^b f(x)\,dx$, to be the common value of L and U. □

We have to introduce a notion of 'integrability' because it can happen that $L \neq U$ as we shall shortly see. It turns out that $L = U$ for a wide enough class of functions to make the above definition useful for our present purposes. In particular, we can show monotonic functions and continuous functions are integrable over any interval. Non-integrable functions tend to be intellectual curiosities, but they are none the less useful for demonstrating the limitations of this definition of the definite integral.

6.11 Examples

(i) $f(x) \equiv C$ constant.
For any dissection D of any interval $[a, b]$ we have

$$M_n = m_n = C$$

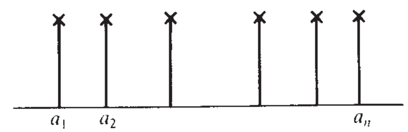
for all n, and therefore

$$U_D = L_D = C(b - a).$$

Hence $f(x)$ is integrable over $[a, b]$ and

$$\int_a^b f(x)\,dx = \int_a^b C\,dx = C(b - a).$$

(ii) $f(x) = 1$ if $x = a_1, a_2, \ldots, a_n$,
$= 0$ otherwise.



For any dissection D of any interval $[a, b]$ we clearly have $L_D = 0$. Therefore $L = 0$. By choosing D to include the points $a_1 \pm \varepsilon$, $a_2 \pm \varepsilon, \ldots, a_n \pm \varepsilon$ we clearly have

$$U_D \leqslant 2n\varepsilon$$

for any $\varepsilon > 0$. Hence $U = 0$ and so $f(x)$ is integrable over $[a, b]$ and $\int_a^b f(x)\,dx = 0$.
(iii) *Dirichlet's function* $f(x) = 1$ if x rational,
$= 0$ otherwise.

For any dissection D of any interval $[a, b]$ we have $M_n = 1$, $m_n = 0$ for all n. (See 1.31.) Therefore $U_D = b - a$, $L_D = 0$. Hence

$$U = b - a > 0 = L$$

and so $f(x)$ is not integrable over $[a, b]$. □

6.12 Exercise

Let $f(x) = x$, $[a, b] = [0, 1]$. Let D_N be the dissection of $[0, 1]$ into N equal subintervals. Show U_{D_N}, L_{D_N} both $\rightarrow \frac{1}{2}$ as $N \rightarrow \infty$ (cf. 6.2).
Deduce $f(x) = x$ is integrable over $[0, 1]$ and that $\int_0^1 x\,dx = \frac{1}{2}$.
Hint: Observe that $L_{D_N} \leqslant L \leqslant U \leqslant U_{D_N}$ and let $N \rightarrow \infty$. □

6.13 Theorem: Riemann's condition

Bounded $f(x)$ is integrable over $[a, b]$ if and only if, given any $\varepsilon > 0$, there exists a dissection D of $[a, b]$ such that

$$U_D - L_D < \varepsilon.$$

Proof Suppose $f(x)$ is bounded integrable over $[a, b]$ and $\varepsilon > 0$ is given. Then we can find dissections D, D' such that

$$L_D > L - \tfrac{1}{2}\varepsilon,$$
$$U_{D'} < U + \tfrac{1}{2}\varepsilon. \qquad \text{(1.34, question 6)}$$

Therefore

$$\begin{aligned} U_{D\cup D'} - L_{D\cup D'} &\leqslant U_{D'} - L_D \qquad \text{(see 6.5)} \\ &< (U + \tfrac{1}{2}\varepsilon) - (L - \tfrac{1}{2}\varepsilon) \\ &= \varepsilon \end{aligned}$$

since $L = U$.
Suppose on the other hand that Riemann's condition is satisfied. Let $\varepsilon > 0$ be given, and let D be chosen in accordance with the condition. Then we have

$$\begin{aligned} 0 &\leqslant U - L \\ &\leqslant U_D - L_D \\ &< \varepsilon; \end{aligned}$$

therefore

$$0 \leqslant U - L < \varepsilon,$$

and hence $L = U$, since ε is arbitrarily small. □

We can use 6.13 to show monotonic functions and continuous functions are integrable as follows.

6.14 Theorem

If $f(x)$ increases over $[a, b]$, then $f(x)$ is integrable over $[a, b]$. □

Proof Let D be the dissection

$$a = x_0 < x_1 < x_2 < \cdots < x_N = b$$

of $[a, b]$. Then we have $M_n = f(x_n)$, $m_n = f(x_{n-1})$ for all n, and

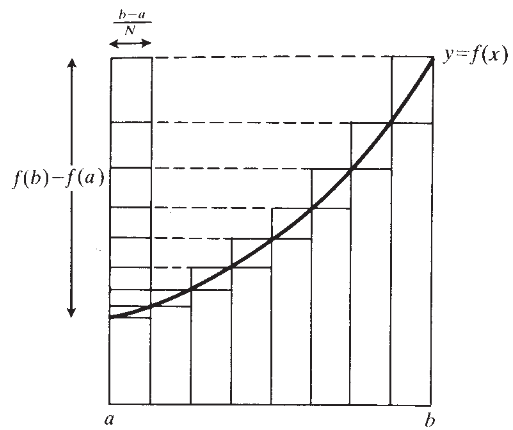


Fig. 6.4

hence

$$\begin{aligned}U_D - L_D &= \sum_1^N (M_n - m_n)(x_n - x_{n-1}) \\ &= \sum_1^N (f(x_n) - f(x_{n-1}))(x_n - x_{n-1}) \\ &= \frac{b-a}{N} \sum_1^N (f(x_n) - f(x_{n-1})),\end{aligned}$$

if we choose D such that $x_n - x_{n-1} = (b-a)/N$ for all n,

$$= \frac{(b-a)(f(b) - f(a))}{N}$$

which is arbitrarily small if N is chosen large enough. □

Observe that $U_D - L_D$ is the total area of the small boxes covering the graph of $f(x)$. These boxes can all be slid horizontally to the left to form a pile of height $f(b) - f(a)$ and width $(b-a)/N$ (see Fig. 6.4).

6.15 Theorem

If $f(x)$ is continuous over $[a, b]$, then $f(x)$ is integrable over $[a, b]$.

Proof Observe firstly that $f(x)$ is bounded over $[a, b]$ by the minimax theorem (4.58).

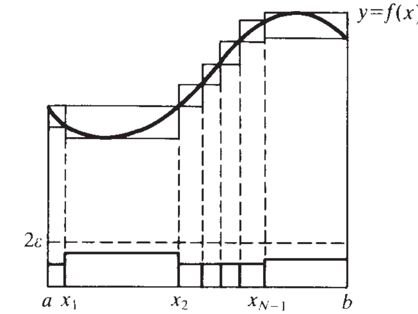


Fig. 6.5

Suppose $\varepsilon > 0$ is given. Construct a dissection D of $[a, b]$ as follows. Let x_1 be the first point in $[a, b]$ such that

$$|f(x_1) - f(a)| = \varepsilon.$$

(See 4.59, question 10.) If no such point exists take $x_1 = b$. If $x_1 < b$, let x_2 be the first point in $[x_1, b]$ such that

$$|f(x_2) - f(x_1)| = \varepsilon.$$

If no such point exists take $x_2 = b$. Define $x_3, x_4, \ldots$ similarly. We must have $x_N = b$ for some N. This is because, if not, then we would have an infinite sequence $(x_n)_{n \geqslant 1}$ such that

$$x_1 < x_2 < \cdots < x_n < \cdots < b$$

and

$$|f(x_n) - f(x_{n+1})| = \varepsilon$$

for all n. Therefore $(x_n)_{n \geqslant 1}$, being increasing and bounded above, would converge to a limit x, say. Hence, by continuity, $f(x_n)$, $f(x_{n+1})$ would both converge to $f(x)$, giving a contradiction.

This dissection D has the property that for every n

$$M_n - m_n < 2\varepsilon.$$

Therefore

$$\begin{aligned}U_D - L_D &= \sum_1^N (M_n - m_n)(x_n - x_{n-1}) \\ &< 2\varepsilon \sum_1^N (x_n - x_{n-1}) \\ &= 2\varepsilon(b-a)\end{aligned}$$

which is arbitrarily small. Hence $f(x)$ is integrable over $[a, b]$ by 6.13. □

Observe that, as in 6.14, $U_D - L_D$ is represented by the total area of the boxes covering the graph of $f(x)$. By sliding these all down to the x-axis we obtain an area included in a rectangle of height 2ε and breadth $b-a$ (see Fig. 6.5).

6.16 Exercise

Which functions are integrable over the intervals stated?

(i) $\operatorname{sgn} x$ over $[-1, 1]$.

(ii) $|x|$ over $[-1, 1]$.

(iii) $f(x) = 1$ if $1 < x < 2$,
$= 0$ otherwise,
over $[0, 3]$. □

The reader will have observed that, while we have given a precise definition of the definite integral $\int_a^b f(x)\,\mathrm{d}x$ for certain functions $f(x)$, we haven't quite related it to $\lim_{\mathrm{d}x\to 0} \Sigma f(x)\,\mathrm{d}x$. This will be our next objective.

6.17 Definition

If D is the dissection

$$a = x_0 < x_1 < x_2 < \cdots < x_N = b$$

of $[a, b]$, then we define the *modulus* of D, denoted by $|D|$, to be

$$|D| = \max_{1\leq n\leq N} (x_n - x_{n-1}).$$ □

6.18 Lemma

If D, D' are two dissections such that $|D| < \delta$ and D' has N' interior points, then

$$U_{D\cup D'} > U_D - \delta N'(M-m)$$

where, as usual,

$$M = \sup_{a\leq x\leq b} f(x),$$

$$m = \inf_{a\leq x\leq b} f(x).$$ □

Proof If D' has one interior point c which occurs in the nth subinterval (x_{n-1}, x_n) of D, then

$$\begin{aligned} U_D - U_{D\cup D'} &= M_n(x_n - x_{n-1}) - M'(c - x_{n-1}) - M''(x_n - c) \\ &\qquad \text{(in the notation of 6.5)} \\ &\leq M(x_n - x_{n-1}) - m(c - x_{n-1}) - m(x_n - c) \\ &= (M-m)(x_n - x_{n-1}) \\ &< (M-m)\delta. \end{aligned}$$

Therefore, if D' has N' interior points,

$$U_D - U_{D\cup D'} < N'(M-m)\delta$$

which gives the result. □

6.19 Theorem

If $f(x)$ is bounded integrable over $[a, b]$, then U_D, L_D both $\to \int_a^b f(x)\,\mathrm{d}x$ as $|D| \to 0$.

Proof Suppose $\varepsilon > 0$ is given. Choose D' such that

$$U_{D'} < U + \tfrac{1}{2}\varepsilon. \qquad \text{(1.34, question 6)}$$

and suppose D' has N' interior points. Choose

$$\delta = \frac{\varepsilon}{2N'(M-m)}.$$

Then, for any D with $|D| < \delta$, we have

$$\begin{aligned} U_D &< U_{D\cup D'} + \delta N'(M-m) \qquad (6.18) \\ &= U_{D\cup D'} + \tfrac{1}{2}\varepsilon \\ &\leq U_{D'} + \tfrac{1}{2}\varepsilon \qquad (6.5) \\ &< U + \varepsilon. \end{aligned}$$

Hence

$$U \leq U_D < U + \varepsilon$$

for all $|D| < \delta$, which shows that $U_D \to U$ as $|D| \to 0$.
Similarly $L_D \to L$ as $|D| \to 0$. □

6.20 Definition

A *Riemann sum* Σ_D for bounded $f(x)$ corresponding to a dissection D of $[a, b]$ is any sum of the form

$$\Sigma_D = \sum_1^N f(c_n)(x_n - x_{n-1})$$

where $c_n \in [x_{n-1}, x_n]$ for each $1 \leq n \leq N$. □

If $f(x)$ is continuous, then U_D, L_D are Riemann sums (by the minimax theorem 4.58), but in general they may not be. Observe that the notation for a Riemann sum should include mention of the points $c_n \in [x_{n-1}, x_n]$. We have chosen to sacrifice a certain amount of precision in the interests of clarity.

6.21 Darboux' theorem

If bounded $f(x)$ is integrable over $[a, b]$, then the Riemann sums $\Sigma_D \to \int_a^b f(x)\,\mathrm{d}x$ as $|D| \to 0$. □

This is the promised rigorization of '$\Sigma f(x)\,\mathrm{d}x \to \int_a^b f(x)\,\mathrm{d}x$ as $\mathrm{d}x \to 0$'.

Proof For any Riemann sum Σ_D corresponding to any dissection D we clearly have

$$L_D \leq \Sigma_D \leq U_D,$$

so the result follows by the sandwich principle (2.16) and 6.19. □

We can now make a start on the fundamental theorem of calculus.

6.22 Theorem

If $f'(x)$ exists and is continuous on $[a, b]$, then

$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a).$$

Proof For any dissection D by

$$a = x_0 < x_1 < \cdots < x_N = b$$

of $[a, b]$, we have

$$\begin{aligned} f(b) - f(a) &= \sum_1^N (f(x_n) - f(x_{n-1})) \\ &= \sum_1^N f'(c_n)(x_n - x_{n-1}), \end{aligned}$$

for some $c_n \in (x_{n-1}, x_n)$ by the mean value theorem (5.16),

$$\to \int_a^b f'(x)\,\mathrm{d}x$$

as $|D| \to 0$ by Darboux' theorem (6.21). Hence the result. □

Theorem 6.22 enables a large class of definite integrals to be evaluated. For example,

$$\int_a^b \mathrm{e}^x\,\mathrm{d}x = \mathrm{e}^b - \mathrm{e}^a,$$

$$\int_a^b \cos x\,\mathrm{d}x = \sin b - \sin a.$$

We shall have to postpone the proof of the other fundamental theorem of calculus—that the derivative of

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is $f(x)$—until we have established some basic properties of the definite integral.

6.23 Linearity of the integral

If $f(x)$, $g(x)$ are continuous over $[a, b]$, and if C is a constant, then

(i) $\int_a^b (f(x) + g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$,

(ii) $\int_a^b Cf(x)\,\mathrm{d}x = C\int_a^b f(x)\,\mathrm{d}x$.

Proof For any dissection D of $[a, b]$ by

$$a = x_0 < x_1 < x_2 < \cdots < x_N = b$$

and $c_n \in [x_{n-1}, x_n]$ for each $1 \le n \le N$, we have

$$\sum_1^N (f(c_n)+g(c_n))(x_n - x_{n-1}) = \sum_1^N f(c_n)(x_n - x_{n-1}) + \sum_1^N g(c_n)(x_n - x_{n-1}),$$

$$\sum_1^N Cf(c_n)(x_n - x_{n-1}) = C\sum_1^N f(c_n)(x_n - x_{n-1}),$$

so the result follows from 6.21 by letting $|D| \to 0$. □

6.24 Integration of inequalities

If $f(x) \le g(x)$ are continuous for all $x \in [a, b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Proof For any dissection D by

$$a = x_0 < x_1 < \cdots < x_N = b$$

and c_n from each subinterval $[x_{n-1}, x_n]$, we have

$$\sum_1^N f(c_n)(x_n - x_{n-1}) \le \sum_1^N g(c_n)(x_n - x_{n-1}),$$

so the result follows by letting $|D| \to 0$. □

6.25 Modular inequality for integrals

If $f(x)$ is continuous on $[a, b]$, then

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Proof For any

$$a = x_0 < x_1 < \cdots < x_N = b$$

and $c_n \in [x_{n-1}, x_n]$, we have

$$\left|\sum_1^N f(c_n)(x_n - x_{n-1})\right| \le \sum_1^N |f(c_n)|(x_n - x_{n-1}),$$

so the result follows by 6.21. □

6.26 Interval additivity

If $a < b < c$ and $f(x)$ is continuous over $[a, c]$, then

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Proof Any dissection D of $[a, c]$ which includes the point b, say

$$a = x_0 < x_1 < \cdots < x_n = b < x_{n+1} < \cdots < x_N = c,$$

generates dissections D', D'' of $[a, b]$, $[b, c]$, namely

$$a = x_0 < x_1 < \cdots < x_n = b,$$
$$b = x_n < x_{n+1} < \cdots < x_N = c.$$

Also, for any $c_r \in [x_{r-1}, x_r]$ for each $1 \le r \le N$, we have

$$\sum_{r=1}^N f(c_r)(x_r - x_{r-1}) = \sum_{r=1}^n f(c_r)(x_r - x_{r-1}) + \sum_{r=n+1}^N f(c_r)(x_r - x_{r-1}),$$

that is

$$\Sigma_D = \Sigma_{D'} + \Sigma_{D''},$$

and if we let $|D| \to 0$, then clearly $|D'|$, $|D''|$ both $\to 0$ too, so the result follows from 6.21. □

6.27 Definition

If $a < b$ and $f(x)$ is continuous over $[a, b]$ we define $\int_b^a f(x)\,dx$ to be $-\int_a^b f(x)\,dx$. □

6.28 Theorem

If $f(x)$ is continuous over $[a, b]$ and x, y, z are any three points in $[a, b]$, then

$$\int_x^y f(t)\,dt + \int_y^z f(t)\,dt = \int_x^z f(t)\,dt.$$ □

Proof is easily achieved by enumerating the various orderings of x, y, z along the line. □

We can now complete our discussion of the fundamental theorem of calculus.

6.29 Theorem

If $f(t)$ is continuous over $[a, b]$ and

$$F(x)=\int_a^x f(t)\,\mathrm{d}t,$$

then $F(x)$ is differentiable over $[a, b]$ and its derivative is $F'(x)=f(x)$. □

Proof If $x\in[a, b]$ and $\varepsilon>0$ are given, then we can choose $\delta>0$ such that

$$|f(t)-f(x)|<\varepsilon$$

for all $t\in[a, b]$ satisfying $|t-x|<\delta$. Therefore, for any $0<h<\delta$, we have

$$\left|\frac{F(x+h)-F(x)}{h}-f(x)\right|=\left|\frac{\int_a^{x+h} f(t)\,\mathrm{d}t-\int_a^x f(t)\,\mathrm{d}t}{h}-f(x)\right|$$

$$=\left|\frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t-f(x)\right| \qquad (6.28)$$

$$=\left|\frac{1}{h}\int_x^{x+h} (f(t)-f(x))\,\mathrm{d}t\right| \qquad (6.11(\mathrm{i}))$$

$$\leqslant\frac{1}{h}\int_x^{x+h} |f(t)-f(x)|\,\mathrm{d}t \qquad (6.25)$$

$$\leqslant\frac{1}{h}\int_x^{x+h} \varepsilon\,\mathrm{d}t \qquad (6.24)$$

$$=\varepsilon.$$

Therefore $F'_+(x)=f(x)$, and $F'_-(x)=f(x)$ is proved similarly (see 5.5). □

6.30 Corollary

Any continuous function $f(x)$ on the interval $[a, b]$ has a primitive on $[a, b]$.

Proof By 6.29 the function

$$F(x)=\int_a^x f(t)\,\mathrm{d}t$$

is a primitive of $f(t)$. □

In applications of integral calculus the problem invariably boils down to finding primitives. In all but a few cases this problem is quite difficult. Various techniques are available for finding primitives of apparently intractable functions. Most important among these are integration by parts and by substitution which we shall now consider.

6.31 Integration by parts

If $f'(x)$, $g'(x)$ exist and are continuous over $[a, b]$, then

$$\int_a^b f'(x)g(x)\,\mathrm{d}x=f(b)g(b)-f(a)g(a)-\int_a^b f(x)g'(x)\,\mathrm{d}x.$$

Proof If $h(x)=f(x)g(x)$ then, by 6.22,

$$\int_a^b h'(x)\,\mathrm{d}x=h(b)-h(a).$$

Therefore

$$\int_a^b (f'(x)g(x)+f(x)g'(x))\,\mathrm{d}x=f(b)g(b)-f(a)g(a), \qquad (5.6)$$

from which the result follows. □

6.32 Worked example

Evaluate $\int_0^1 x\mathrm{e}^x\,\mathrm{d}x$.

Let $f'(x)=\mathrm{e}^x$, $g(x)=x$. Then we can take $f(x)=\mathrm{e}^x$ and we have $g'(x)=1$. Therefore, by 6.31,

$$\int_0^1 x\mathrm{e}^x\,\mathrm{d}x=x\mathrm{e}^x|_0^1-\int_0^1 \mathrm{e}^x\,\mathrm{d}x$$

$$=\mathrm{e}-\mathrm{e}^x|_0^1$$

$$=\mathrm{e}-(\mathrm{e}-1)$$

$$=1.$$

□

Observe that we are using the notation

$$f(x)|_a^b=f(b)-f(a).$$

6.33 Exercises

Evaluate the following integrals.

(i) $\int_0^{\pi} x\cos x\,dx$ (ii) $\int_0^1 x^2 e^x\,dx$

(iii) $\int_0^{\pi} e^x \sin x\,dx$ (iv) $\int_1^2 \log x\,dx$ □

6.34 Integration by substitution

If $f(x)$ is continuous over $[a, b]$ and if $g'(t)$ exists and is continuous over $[\alpha, \beta]$, where $g(\alpha)=a$, $g(\beta)=b$, then

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(g(t))g'(t)\,dt.$$

Proof Let $F(x)$ be a primitive of $f(x)$ (see 6.30). If $h(t)=F(g(t))$ then, by 6.22,

$$\int_\alpha^\beta h'(t)\,dt = h(\beta)-h(\alpha).$$

Therefore

$$\begin{aligned}\int_\alpha^\beta F'(g(t))g'(t)\,dt &= F(g(\beta))-F(g(\alpha)) \quad (5.10)\\ &= F(b)-F(a)\\ &= \int_a^b F'(x)\,dx, \quad (6.22)\end{aligned}$$

that is

$$\int_\alpha^\beta f(g(t))g'(t)\,dt = \int_a^b f(x)\,dx,$$

which is the required result. □

6.35 Worked examples

(i) Evaluate $\int_0^1 \frac{dx}{1+x^2}$.

Let $f(x)=1/(1+x^2)$, $g(t)=\tan t$. Then $g'(t)=\sec^2 t$ and therefore, by 6.34,

$$\begin{aligned}\int_0^1 \frac{dx}{1+x^2} &= \int_0^{\pi/4} \frac{1}{1+\tan^2 t}\sec^2 t\,dt\\ &= \int_0^{\pi/4} dt\\ &= \pi/4.\end{aligned}$$

(ii) Evaluate $\int_0^1 e^{2t}\,dt$.

Let $f(x)=e^x$, $g(t)=2t$. Then $g'(t)=2$ and therefore, by 6.34,

$$\int_0^1 e^{2t}\,.\,2\,dt = \int_0^2 e^x\,dx,$$

and hence

$$\begin{aligned}\int_0^1 e^{2t}\,dt &= \tfrac{1}{2}\int_0^2 e^x\,dx\\ &= \tfrac{1}{2}e^x\big|_0^2\\ &= \tfrac{1}{2}(e^2-1).\end{aligned}$$ □

6.36 Exercises

Evaluate the following integrals.

(i) $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)}}$ (ii) $\int_0^1 \frac{dx}{1+\sqrt{x}}$

(iii) $\int_0^{\frac{1}{2}\pi} \sin(t+\tfrac{1}{2}\pi)\,dt$ (iv) $\int_0^1 \frac{dt}{1+\sqrt{t}}$

Hints: Make the substitutions $x=\sin t$, t^2, $t+\frac{1}{2}\pi$, $\sqrt{t}$. □

There are certain definite integrals which cannot be evaluated in closed form because their integrands (functions to be integrated) have no primitives expressible in terms of the elementary functions. Examples of such integrals are $\int_0^1 e^{x^2}\,dx$, $\int_0^\pi \sin x/x\,dx$. To evaluate these integrals one would have to resort to numerical methods, and be content with an approximate answer. Of course, e.g., $f(x)=e^{x^2}$ does have a primitive, namely

$$F(x)=\int_0^x e^{t^2}\,dt,$$

but $F(x)$ cannot be expressed in a simpler form.

6.37 Miscellaneous exercises

1. Discuss the integrability over $[0, 1]$ of the following functions.

(i) $\sin\frac{1}{x}$ (ii) $x\sin\frac{1}{x}$

(Set each equal to 0 at $x=0$.)

2. Prove the following inequalities.

(i) $\frac{1}{2}\pi \leqslant \int_0^{\pi} \frac{\sin x}{x}\,\mathrm{d}x \leqslant \pi.$

(ii) $\frac{2}{3} \leqslant \int_0^1 \mathrm{e}^{-x^2}\,\mathrm{d}x \leqslant \frac{21}{30}.$

Hints: Integrate the inequalities

(i) $\frac{\pi - x}{\pi} \leqslant \frac{\sin x}{x} \leqslant 1,$ (5.44, question 3)

(ii) $1 - x^2 \leqslant \mathrm{e}^{-x^2} \leqslant 1 - x^2 + \frac{x^4}{2}.$

3. Prove that if $f'(x)$ exists and is continuous over $[a, b]$, then

$$\int_a^b f(x) \sin nx\,\mathrm{d}x \to 0$$

as $n \to \infty$. *Hint:* Integrate by parts.

4. Prove that, if $f(t)$ is bounded integrable over $[a, x]$ for every $x \in [a, b]$, then

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is continuous, but may fail to be differentiable if $f(t)$ is discontinuous. *Hint:* Consider $f(t) = \operatorname{sgn} t$, $[a, b] = [-1, 1]$.

5. Prove that, if $f(x)$ is infinitely differentiable over $[a, x]$, then

$$f(x) = f(a) + (x-a)f'(a) + \cdots + \frac{(x-a)^N}{N!} f^{(N)}(a) + R_N(x),$$

where

$$R_N(x) = \frac{1}{N!}\int_a^x (x-t)^N f^{(N+1)}(t)\,\mathrm{d}t.$$

(Taylor's theorem with integral form of remainder, cf. 5.41) *Hint:* Integrate $R_N(x)$ by parts N times.

6. Prove that if $f(x)$, $g(x)$ are continuous and $m \leqslant f(x) \leqslant M$, $g(x) \geqslant 0$ for all $x \in [a, b]$, then

$$m\int_a^b g(x)\,\mathrm{d}x \leqslant \int_a^b f(x)g(x)\,\mathrm{d}x \leqslant M\int_a^b g(x)\,\mathrm{d}x.$$

Deduce that

$$\int_a^b f(x)g(x)\,\mathrm{d}x = f(c)\int_a^b g(x)\,\mathrm{d}x$$

for some $c \in [a, b]$. (First mean value theorem for integrals.) *Hint:* Use 4.24.

7. Prove that if $f(x)$, $g(x)$ are continuous and

$$m \leqslant \int_a^x f(t)\,\mathrm{d}t \leqslant M,$$

$g(x) \geqslant 0$, $g'(x)$ exists and is continuous and $\leqslant 0$ for all $x \in [a, b]$, then

$$mg(a) \leqslant \int_a^b f(x)g(x)\,\mathrm{d}x \leqslant Mg(a)$$

(second mean value theorem for integrals).

Prove that, for any $0 < a < b$,

$$\left|\int_a^b \frac{\sin x}{x}\,\mathrm{d}x\right| \leqslant \frac{2}{a}.$$

8. Prove that, for any continuous $f(x)$, $g(x)$,

(i) $\int_a^b |f(x)+g(x)|\,\mathrm{d}x \leqslant \int_a^b |f(x)|\,\mathrm{d}x + \int_a^b |g(x)|\,\mathrm{d}x,$

(ii) $\left(\int_a^b f(x)g(x)\,\mathrm{d}x\right)^2 \leqslant \int_a^b (f(x))^2\,\mathrm{d}x \int_a^b (g(x))^2\,\mathrm{d}x.$

Hints: For (i) use 6.23 and 6.24. For (ii) consider $\int_a^b (f(x) + \lambda g(x))^2\,\mathrm{d}x$, where λ is a parameter.

9. Prove that, if $f(x)$ is continuous and $f(x) \geqslant 0$ for all $x \in [a, b]$, and if $f(c) > 0$ for some $c \in [a, b]$, then

$$\int_a^b f(x)\,\mathrm{d}x > 0.$$

Show by counter-example that we cannot drop the assumption that $f(x)$ is continuous.

10. By considering Riemann sums for suitable integrals, find the limits of the sequences whose nth terms are the following.

(i) $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$

(ii) $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n}$

(iii) $\sqrt[n]{\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]}$