16. Given that $\sum_1^\infty a_n$ converges, prove that
(i) for any $N \geq 1$, the series $\sum_{N+1}^\infty a_n$ converges,
(ii) the sequence $(\sum_{N+1}^\infty a_n)_{N\geq 1}$ is null.
17. Show that if the power series $\sum_0^\infty a_n x^n$ has radius of convergence R, then the formally differentiated series $\sum_0^\infty n a_n x^{n-1}$ also has radius of convergence R.

4 Continuous functions

Having discussed convergence of sequences and series in Chapters 2 and 3, we now move to convergence of functions of a continuous variable. We shall need a theory of continuous convergence to give an adequate description of the process of differentiating a function (see Chapter 5). A theory of continuous convergence also enables a proper definition of the concept of a continuous function to be made. Since properties of continuous functions underlie important theorems in differential and integral calculus, it seems opportune to devote a chapter to continuous functions before proceeding to calculus.

A sequence can be thought of as a function of an integer variable n, and the problem of convergence is concerned with what happens as n gets large or 'tends to infinity'. With a function $f(x)$ of a continuous variable x, the question is what happens as x 'tends to' a particular value c. If $f(x)$ approaches a definite value l as x approaches c, we say $f(x)$ *tends to* l as x *tends to* c, and we write $f(x) \to l$ as $x \to c$. We call l the *limit* of $f(x)$ as x tends to c, and write $l = \lim_{x\to c} f(x)$. For example, $x^2 \to 4$ as $x \to 2$, equivalently, $\lim_{x\to 2} x^2 = 4$.

The arrow notation can also be used in the context of sequences and series. If the sequence $(a_n)_{n\geq 1}$ converges to a, we write $a_n \to a$ as $n \to \infty$, also $\lim_{n\to\infty} a_n = a$. If the series $\sum_1^\infty a_n$ converges, we can write $\sum_1^\infty a_n = \lim_{N\to\infty} \sum_1^N a_n$.

If we try to analyse what we mean by saying $f(x) \to l$ as $x \to c$, we find ourselves again in a two-handed game situation, just as we did for sequential convergence. Here we must be able to say that $f(x)$ is as near to l as *someone else* might want, provided we take x near enough to c. The rigorous definition traditionally makes use of the Greek letters ε, δ and is as follows (see Fig. 4.1).

4.1 Definition

We say $f(x) \to l$ as $x \to c$ if, for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - l| < \varepsilon$$

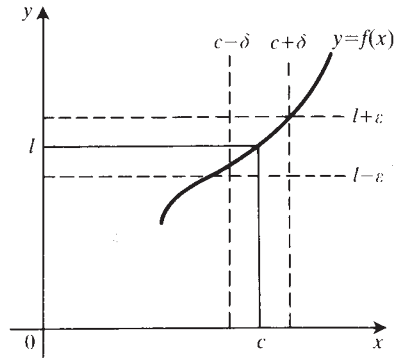


Fig. 4.1

for all x satisfying

$$0<|x-c|<\delta.$$ □

Observe that, for the purposes of the present game, we require an ε-player and a δ-player. As usual the ε-player goes first, and the limit exists if the δ-player can come up with a strategy which copes with any play the ε-player may make.

Observe also that we are not interested in what happens when $x=c$. We want to know what happens as x *tends* to c without actually ever getting there.

4.2 Example

$x^2 \to 4$ as $x \to 2$ (see Fig. 4.2).

Suppose $\varepsilon>0$ is given. We have to indicate how $\delta>0$ can be

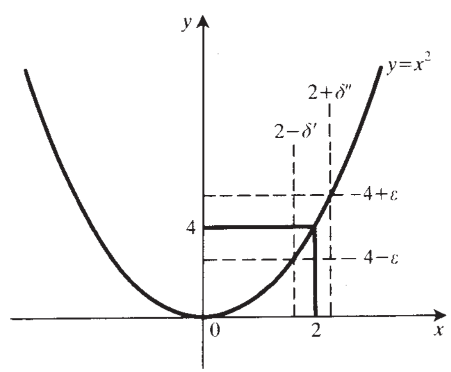


Fig. 4.2

chosen such that

$$|x^2-4|<\varepsilon$$

whenever

$$0<|x-2|<\delta.$$

Let $2-\delta'=\sqrt{(4-\varepsilon)}$, $2+\delta''=\sqrt{(4+\varepsilon)}$ and take $\delta=\min\{\delta', \delta''\}$. Then $\delta>0$ and, for any x satisfying

$$0<|x-2|<\delta,$$

we clearly have

$$|x^2-4|<\varepsilon.$$ □

A theory of limits for functions can now be built up in an analogous fashion to that for sequences. We can either translate the methods to the new situation, or we can use results from the sequential theory. The tie-up is provided by the following theorem.

4.3 Theorem

$f(x)\to l$ as $x\to c$ if and only if, for any sequence $(x_n)_{n\geqslant 1}$ such that $x_n \neq c$ for all n and $x_n\to c$ as $n\to\infty$, we have $f(x_n)\to l$ as $n\to\infty$.

Proof Suppose firstly that $f(x)\to l$ as $x\to c$, and that we have a sequence $x_n\neq c$ which $\to c$ as $n\to\infty$. We have to show $f(x_n)\to l$ as $n\to\infty$, so we must allow ourselves to be given $\varepsilon>0$, and then indicate how we might choose a positive integer N such that

$$|f(x_n)-l|<\varepsilon$$

for all $n>N$. The fact that $f(x)\to l$ as $x\to c$ means we can choose $\delta>0$ such that

$$|f(x)-l|<\varepsilon$$

whenever

$$0<|x-c|<\delta$$

(by 4.1). And the fact that $x_n\to c$ as $n\to\infty$, and $x_n\neq c$ for all n means we can choose N such that

$$0<|x_n-c|<\delta$$

for all $n>N$. Hence N has the required properties.

Now suppose conversely that we are told that $f(x_n)\to l$ whenever $(x_n)_{n\geqslant 1}$ is a sequence which converges to c but is never equal to c. We want to show this implies $f(x)\to l$ as $x\to c$. We shall achieve this by showing that, if $f(x)$ *doesn't* $\to l$ as $x \to c$, then we can construct a sequence $x_n \neq c$ converging to c such that $f(x_n)$ doesn't converge to $f(l)$. Failure of $f(x)$ to tend to l means that the ε-player can find ε for which the δ-player has no suitable δ. Therefore, for every positive integer n, playing $\delta = 1/n$ is inadequate to cope with this ε, so there must exist a real number, which we can call x_n, such that

$$0 < |x_n - c| < \frac{1}{n}$$

but

$$|f(x_n) - l| \geq \varepsilon.$$

This constructs a sequence (x_n) with the required properties. □

4.4 Application

We can use 4.3 to give an alternative proof of 4.2.

In fact, 2.11 gives immediately that, for any sequence $x_n \neq 2$ which $\to 2$, we have $x_n^2 \to 4$.

4.5 Theorem: Arithmetic of limits

If $f(x) \to l$ and $g(x) \to m$ as $x \to c$, then

(i) $f(x) + g(x) \to l + m$,

(ii) $f(g)g(x) \to lm$,

(iii) $f(x)/g(x) \to l/m$,

as $x \to c$, provided (for (iii)) $m \neq 0$ and $g(x) \neq 0$ for all $x \neq c$.

Proof (i) If $(x_n)_{n \geq 1}$ is any sequence such that $x_n \to c$ and $x_n \neq c$, then $f(x_n) \to l$ and $g(x_n) \to m$ by 4.3, and therefore, by 2.11, $f(x_n) + g(x_n) \to l + m$.

(ii) and (iii) are proved similarly. □

4.6 Corollaries

(i) If $f(x) \to l$ as $x \to c$, and C is a constant, then $Cf(x) \to Cl$ as $x \to c$.

(ii) If $f(x) \to l$ and $g(x) \to m$ as $x \to c$, then

$$f(x) - g(x) \to l - m$$

as $x \to c$. □

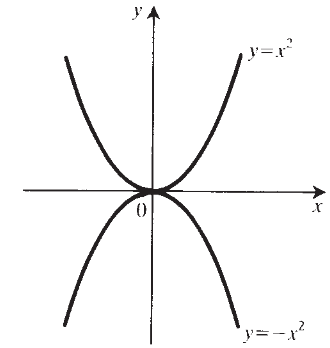


Fig. 4.3

4.7 Theorem: Taking limits in inequalities

If $f(x) \to l$, $g(x) \to m$ as $x \to c$, and if $f(x) \leq g(x)$ for all $x \neq c$, then $l \leq m$.

Proof Use 2.14 and 4.3. □

Just as for sequences, one cannot assume that a *strict* inequality persists in the limit. For example, $x^2 > -x^2$ for all $x \neq 0$, but $\pm x^2$ both $\to 0$ as $x \to 0$ (Fig. 4.3).

4.8 Theorem: Sandwich principle

If $f(x) \leq g(x) \leq h(x)$ for all $x \neq c$, and if $f(x)$, $h(x)$ both $\to l$ as $x \to c$, then also $g(x) \to l$ as $x \to c$.

Proof Follows from the sandwich principle for sequences (2.16) and 4.3. □

It is sometimes necessary to consider one-sided limits. For example, $f(x) = \sqrt{x}$ can only be defined for $x \geq 0$, so it would only be possible to consider the limit as $x \to 0$ through *positive* values. Accordingly, we make the following definition.

4.9 Definition

We say $f(x)$ tends to l as x tends to c *from the right*, and we write $f(x) \to l$ as $x \to c_+$ (also $l = \lim_{x \to c_+} f(x)$) if, for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - l| < \varepsilon$$

for all x satisfying

$$c < x < c + \delta.$$

4.10 Example

$\sqrt{x} \to 0$ as $x \to 0_+$.

In fact, if $\varepsilon > 0$ is given, then, for $\delta = \varepsilon^2$, we have $\sqrt{x} < \varepsilon$ whenever $0 < x < \delta$. □

We define limits on the left, with the notation $x \to c_-$, similarly.

Clearly $\lim_{x \to c} f(x) = l$ if and only if $\lim_{x \to c_+} f(x) = \lim_{x \to c_-} f(x) = l$. It is conceivable that the limits from the left and right may differ. Consider the following example.

4.11 Example: The sign function

We define the *sign* function, which we abbreviate to $\operatorname{sgn} x$, as follows (see Fig. 4.4):

$$\begin{aligned} \operatorname{sgn} x &= 1 \quad \text{if } x > 0, \\ &= 0 \quad \text{if } x = 0, \\ &= -1 \quad \text{if } x < 0. \end{aligned}$$

If $f(x) = \operatorname{sgn} x$, we have $\lim_{x \to 0_+} f(x) = 1$, $\lim_{x \to 0_-} f(x) = -1$. It follows that $\lim_{x \to 0} f(x)$ cannot exist in this case. □

This is our first instance of non-existence of a limit. Other ways in which a limit may fail to exist are illustrated in the following examples.

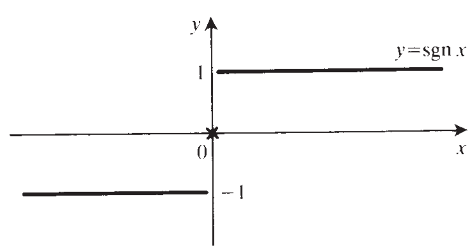


Fig. 4.4

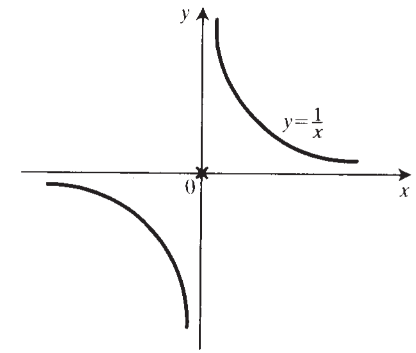


Fig. 4.5

4.12 Example

Let $f(x)$ be defined thus (see Fig. 4.5):

$$\begin{aligned} f(x) &= \frac{1}{x} \quad \text{if } x \neq 0, \\ &= 0 \quad \text{if } x = 0. \end{aligned}$$

Observe that the formula $1/x$ doesn't define a value at $x = 0$. If we want our function $f(x)$ to have a value at $x = 0$, we must make a special definition for this point. We have chosen to define $f(0) = 0$, though we could have taken any other value if we had liked.

$\lim_{x \to 0} f(x)$ doesn't exist for this function since, e.g., if $x_n = 1/n$, then the sequence $f(x_n)$ diverges to infinity. Also, e.g., for $x_n = -1/n$, the sequence $f(x_n)$ diverges to minus infinity. □

The above example leads us to make the following definition.

4.13 Definition

We say $f(x)$ *tends to infinity* as x tends to c from the right, and we write $f(x) \to \infty$ as $x \to c_+$ (also $\lim_{x \to c_+} f(x) = \infty$) if, given any $C > 0$, there exists $\delta > 0$ such that $f(x) > C$ for all x satisfying $c < x < c + \delta$. □

Observe that the two-handed game here involves a C-player and a δ-player. Clearly $1/x \to \infty$ as $x \to 0_+$. Clearly also, there is an analogue of 4.3 which says that, in general, $f(x)\to\infty$ as $x\to c_+$ if and only if, for any sequence $x_n>c$ which converges to c, the sequence $f(x_n)$ diverges to infinity.

One can similarly define $f(x)\to\infty$ as $x\to c_-$, or as $x\to c$, also $f(x)\to-\infty$ as $x\to c$ from left or right or both sides simultaneously. For example, $1/x\to-\infty$ as $x\to 0_-$. □

It is important to emphasize that we do *not* regard ∞ as a number. When we write $\lim_{x\to c} f(x)=\infty$, we mean $f(x)$ *diverges* to infinity as $x\to c$. The usage $\lim_{x\to c} f(x)=\infty$ is of course an abuse of notation, since in this situation $f(x)$ has *no limit* as $x\to c$. It is nevertheless a convenient usage in many contexts and, properly understood, should not lead to any confusion.

Another example of non-existence of a limit is the following.

4.14 Example

Let $f(x)$ (see Fig. 4.6) be defined by saying

$$f(x)=\sin\frac{1}{x} \quad \text{if } x\neq 0,$$
$$=0 \quad \text{if } x=0.$$

$f(x)$ has no limit as $x\to 0_+$ since, if we let $x_n=1/(n+\frac{1}{2})\pi$, then we have $f(x_n)=(-1)^n$, which is an oscillating sequence. Obviously $\lim_{x\to 0_-} f(x)$ doesn't exist either for similar reasons. □

Please note that officially we have not yet defined the function $\sin x$. We shall nevertheless take the liberty of using $\sin x$ in examples and illustrations on the understanding that it will in due course be given a rigorous definition and its properties rigorously established. (See 4.45 *et seq.*)

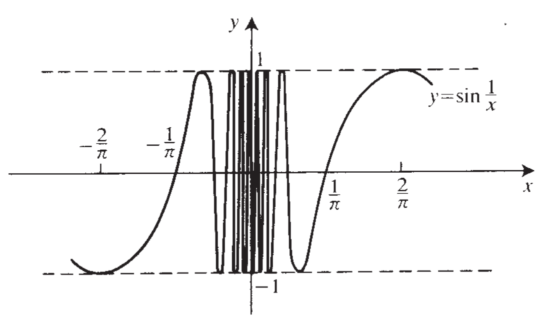


Fig. 4.6

One more type of limit remains to be considered, and that is where the *variable* diverges to infinity. The following two definitions are typical.

4.15 Definitions

(i) We say $f(x)$ tends to l as x tends to infinity, and we write $f(x)\to l$ as $x\to\infty$, or $\lim_{x\to\infty} f(x)=l$, if, given any $\varepsilon>0$, there exists $C>0$ such that

$$|f(x)-l|<\varepsilon$$

for all $x>C$.

(ii) We say $f(x)$ tends to infinity as x tends to infinity, and we write $f(x)\to\infty$ as $x\to\infty$, or $\lim_{x\to\infty} f(x)=\infty$, if, given any $C>0$, there exists $C'>0$ such that $f(x)>C$ for all $x>C'$. □

Definitions for $x\to-\infty$ are similar.

4.16 Examples

(i) $f(x)=1/(1+x^2)$ (Fig. 4.7). Clearly $f(x)\to 0$ as $x\to\pm\infty$ here.

(ii) $f(x)=x^3$ (Fig. 4.8). Clearly $f(x)\to\pm\infty$ as $x\to\pm\infty$ in this case. □

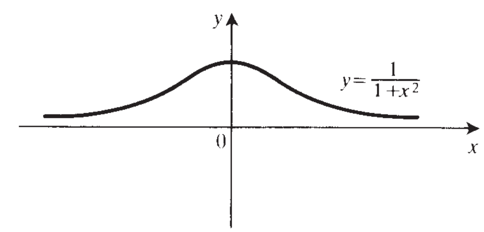


Fig. 4.7

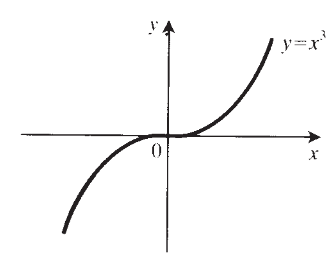


Fig. 4.8

The analogue of 4.3 for limits as $x \to \infty$ is that $f(x) \to l$ as $x \to \infty$ if and only if $f(x_n) \to l$ as $n \to \infty$ for any sequence x_n which $\to \infty$ as $n \to \infty$. Theorems 4.5 to 4.8 are all true for limits as $x \to \infty$ with the same proofs. These remarks also apply to limits as $x \to -\infty$.

4.17 Exercises

Investigate the existence or otherwise of the following limits.

(i) $\lim_{x\to 0} |\operatorname{sgn} x|$ (ii) $\lim_{x\to 0} x \sin \frac{1}{x}$ (iii) $\lim_{x\to 0} \frac{1}{x} \sin \frac{1}{x}$

Draw graphs in each case. □

Such is the theory of continuous limits. We now turn to continuous functions.

By a continuous function we mean one which has a continuous graph, i.e., the graph is a single unbroken curve which can be drawn without lifting pen from paper, as in Fig. 4.9.

Among the functions we have considered so far, it is clear that x^2, x^3, $1/(1+x^2)$ are continuous, but that $\operatorname{sgn} x$, $1/x$, $\sin 1/x$ are not.

It is the discontinuous functions which provide the clue as to how the property of being continuous should be rigorously defined. For example, the graph of $\operatorname{sgn} x$ has a break at $x = 0$, and this is because $\lim_{x\to 0_+} \operatorname{sgn} x = 1$, $\lim_{x\to 0_-} \operatorname{sgn} x = -1$ but $\operatorname{sgn} 0 = 0$. The functions $1/x$, $\sin 1/x$ have discontinuities at $x = 0$ because they have no limits there. This suggests that the condition for continuity should be that the function should converge to its value at every point. Discontinuous functions are those which converge to the wrong value, or converge to no value at all.

We therefore make the following definition.

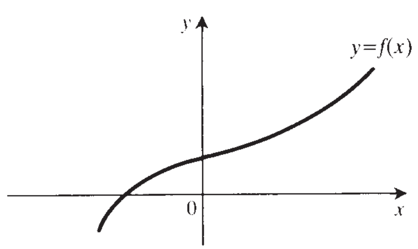


Fig. 4.9

4.18 Definition

The function $f(x)$ is *continuous* at the point $x = c$ if $\lim_{x\to c} f(x) = f(c)$. □

It is easy to prove on the basis of this definition that x^2, x^3, $1/(1+x^2)$ are continuous at every point, and that $\operatorname{sgn} x$ and $1/x$ are continuous everywhere except at $x = 0$. The function $\sin 1/x$ is clearly discontinuous at $x = 0$, and we shall shortly be able to prove it is continuous at all other points.

We define one-sided continuity by saying e.g. $f(x)$ is continuous *on the right* at $x = c$ if $\lim_{x\to c_+} f(x) = f(c)$. For example, $f(x) = \sqrt{x}$ is continuous on the right at $x = 0$ since $\lim_{x\to 0_+} \sqrt{x} = 0 = \sqrt{0}$. (See 4.10.) In fact, the function $\sqrt{x}$ is continuous at every $x > 0$ as well because of the inequality

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}$$

for all $x > 0$, $y > 0$. (See 2.39, question 3.)

We must now address ourselves to the question as to which functions are continuous. Also, having built up a stock of continuous functions, we must then ask what we can do with them.

The simplest way to manufacture continuous functions is by means of the operations of arithmetic, in which connection we have the following theorem.

4.19 Theorem

If $f(x)$, $g(x)$ are continuous at $x = c$, then so are
(i) $f(x) + g(x)$,
(ii) $f(x)g(x)$,
(iii) $f(x)/g(x)$,
provided, in case (iii), $g(c) \neq 0$.

Proof is immediate from 4.5. The condition $g(c) \neq 0$ ensures that $g(x) \neq 0$ for all x belonging to some open interval (a, b) containing c. This is because there must be a $\delta > 0$ such that

$$|g(x) - g(c)| < |g(c)|$$

for all x satisfying $|x - c| < \delta$. (Play $\varepsilon = |g(c)|$.) Hence $g(x) \neq 0$ for all $x \in (c - \delta, c + \delta)$.

It follows that $f(x)/g(x)$ is defined for all $x \in (c - \delta, c + \delta)$, and it is clear that, for the purposes of evaluating $\lim_{x\to c} f(x)/g(x)$, it is enough to consider what happens on this interval. □

4.20 Corollaries

If $f(x)$, $g(x)$ are continuous at $x = c$, then so are
(i) $f(x) - g(x)$,
(ii) $Cf(x)$,
where C is any constant. □

4.21 Applications

(i) Any polynomial, e.g.,

$$p(x) = 5x^7 - 2x^6 + x^2 - 3$$

is continuous for every x.

In fact, $f(x) = \text{constant}$, $g(x) = x$ are clearly continuous from the definition (4.18), and $p(x)$ is obtained from $f(x)$, $g(x)$ by addition and multiplication.

(ii) Any rational function $p(x)/q(x)$ where $p(x)$, $q(x)$ are polynomials, e.g.

$$\frac{p(x)}{q(x)} = \frac{x^3 + 2x^2 + 3x + 4}{5x^3 + 6x^2 + 7x + 8},$$

is continuous everywhere except where the denominator $q(x)$ vanishes. □

Another way to combine continuous functions is by *composition*, or forming *composites*, defined as follows.

4.22 Definition

The *composition* or *composite* of two functions $f(x)$, $g(x)$ is the function $h(x) = f(g(x))$. □

For example (Fig. 4.10), if $f(x) = 1 + x$, $g(x) = |x|$, then

$$f(g(x)) = 1 + |x|,$$
$$g(f(x)) = |1 + x|.$$

Observe that $f(g(x))$, $g(f(x))$ are different functions in this case.

4.23 Theorem

If $g(x)$ is continuous at $x = c$, and $f(y)$ is continuous at $y = g(c)$, then $f(g(x))$ is continuous at $x = c$.

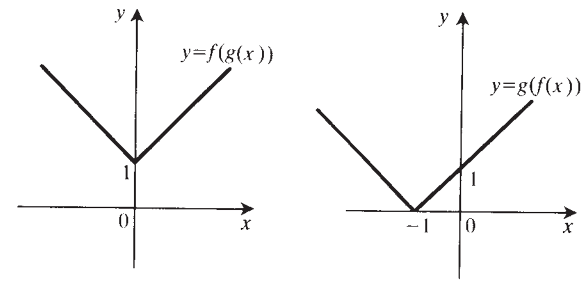


Fig. 4.10

Proof Let (x_n) be any sequence which converges to c. Then $g(x_n)$ converges to $g(c)$ by the continuity of $g(x)$ at $x = c$, and therefore $f(g(x_n))$ converges to $f(g(c))$ by the continuity of $f(y)$ at $y = g(c)$. □

Hence, for example, the functions $1 + |x|$, $|1 + x|$ are continuous for all x. The function $|x|$ is of course continuous for all x since, e.g., $|x_n| \to |c|$ whenever $x_n \to c$. (See 2.39, question 2.)

The next method we shall consider of constructing new continuous functions from old ones is by taking *inverses*. Consider the function $f(x) = 3x + 4$. If we write

$$y = 3x + 4,$$

then we can solve for x in terms of y to obtain

$$x = \tfrac{1}{3}(y - 4).$$

We shall call the function giving x in terms of y the *inverse* function of $f(x)$ and denote it by $f^{-1}(y)$, so that we have

$$f^{-1}(y) = \tfrac{1}{3}(y - 4).$$

Observe that $f^{-1}(y) = x$ if and only if $f(x) = y$.

Constructing inverse functions is not always as easy as in the above example. This is because the equation $y = f(x)$ does not in general have a *unique* solution in x for *every* given y. Consider instead the example $f(x) = x^2$. If we try to solve $y = x^2$ in this case we find that, for $y > 0$, there are two solutions, namely $x = \pm\sqrt{y}$, and, for $y < 0$, there are no solutions for x. If we want to define an inverse function $f^{-1}(y)$ for $f(x) = x^2$, we shall have to restrict its domain of definition to $y \geq 0$ and, for $y > 0$, decide which square root we shall take. The usual convention is to take $f^{-1}(y)$ equal to the positive square root of y, which we denote by $\sqrt{y}$. The negative square root is then of course denoted by $-\sqrt{y}$.

For a general function $f(x)$, we ensure *existence* of a solution x of the equation $y = f(x)$ for given y by means of the intermediate value theorem (see 4.24), and we ensure *uniqueness* of the solution by requiring $f(x)$ to be 'strictly monotonic' (see 4.25).

4.24 Intermediate value theorem

If $f(x)$ is continuous at every x in the closed interval $[a, b]$, and if γ satisfies $f(a) < \gamma < f(b)$, then there exists $c \in (a, b)$ such that $f(c) = \gamma$. □

N.B. If $f(x)$ is defined only for $x \in [a, b]$ we assume continuity on the right at a and on the left at b.

The intermediate value theorem is of course highly plausible—some might say downright obvious. Consider, however, the case $f(x) = x^2$, $[a, b] = [1, 2]$, $\gamma = 2$. We must have $c = \sqrt{2}$, which is of

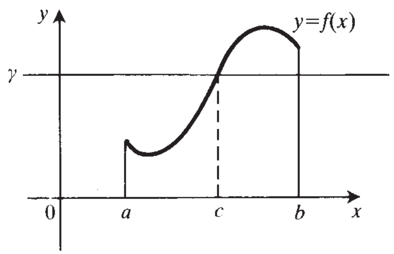


Fig. 4.11

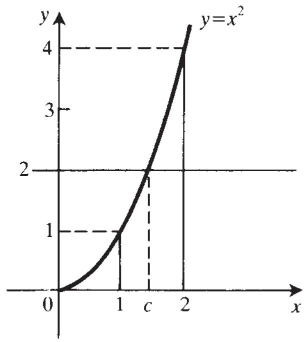


Fig. 4.12

course irrational. It follows that the intermediate value theorem is *false* in the system of rational numbers. It also follows that, to prove the theorem in the real number system, we should expect to have to appeal to the upper bound axiom, as indeed we shall.

Proof of 4.24 Let E be the set

$$E = \{x \in [a, b] : f(x) < \gamma\}.$$

E is non-empty since $a \in E$, and is bounded above since b is an upper bound. Therefore, by the upper bound axiom, E has a supremum. Let $c = \sup E$. We shall show $f(c) = \gamma$.

Choose sequences $x_n \in E$, $y_n \notin E$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = c.$$

(See 2.39, question 8.) Then, since $f(x)$ is continuous at $x = c$, we have

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n) = f(c).$$

But, since $x_n \in E$, $y_n \notin E$, we also have

$$f(x_n) < \gamma \leq f(y_n).$$

Therefore by the sandwich principle (2.16), we obtain $f(c) = \gamma$ as required. □

We should note that it is clear that $c \in [a, b]$ since $a \in E$ and b is an upper bound of E. It is clear also that we can assume $y_n \in [a, b]$ since, if $c < b$ there is no problem, and if $c = b$ then $c \notin E$, so we can take $y_n = b = c$ for all n.

4.25 Definition

We say $f(x)$ is *strictly increasing* over the interval $[a, b]$ if $f(x) < f(y)$ for every $x < y$ both $\in [a, b]$. We say $f(x)$ is *strictly decreasing* if $f(x) > f(y)$ whenever $x < y$. □

For example, $f(x) = \sin x$ is strictly increasing over $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$, and strictly decreasing over $[\frac{1}{2}\pi, \frac{3}{2}\pi]$ (Fig. 4.13).

We say $f(x)$ is *strictly monotonic* over $[a, b]$ if *either* $f(x)$ strictly increases over $[a, b]$ *or* $f(x)$ strictly decreases over $[a, b]$.

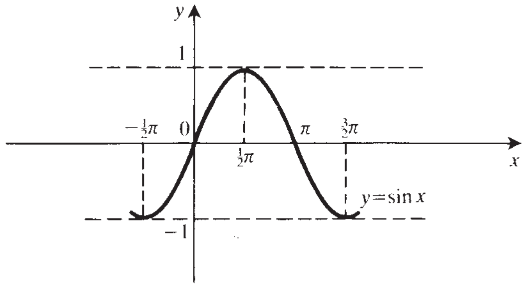


Fig. 4.13

4.26 Inverse function theorem

If $f(x)$ is continuous and strictly increasing over an interval $[a, b]$, then an inverse function $f^{-1}(y)$, such that $f^{-1}(y) = x$ whenever $f(x) = y$, can be defined for all $y \in [\alpha, \beta]$, where $\alpha = f(a)$, $\beta = f(b)$. Also, $f^{-1}(y)$ is continuous for every $y \in [\alpha, \beta]$.

Proof For every $y \in [\alpha, \beta]$, existence of x such that $f(x) = y$ follows from the intermediate value theorem (4.24), and uniqueness of this x follows from the fact that $f(x)$ is strictly increasing. Therefore the inverse function $f^{-1}(y)$ is certainly defined. It only remains to show that $f^{-1}(y)$ is continuous.

Suppose the sequence (y_n) converges to y. We must show that $f^{-1}(y_n)$ converges to $f^{-1}(y)$. Let $x_n = f^{-1}(y_n)$, $x = f^{-1}(y)$. We shall show firstly that (x_n) must converge, and then secondly that (x_n) must converge to x.

Suppose (x_n) were to diverge. Then, since (x_n) is bounded, it must have two subsequences $(x_{m_r})_{r\geq 1}$, $(x_{n_r})_{r\geq 1}$ which converge to

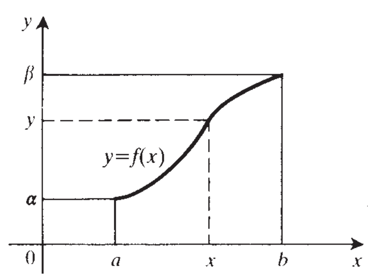


Fig. 4.14

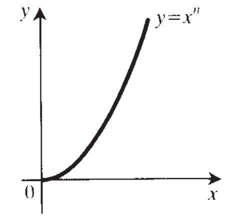


Fig. 4.15

distinct limits $l \neq l'$. (See 2.39, question 10.) Therefore, by continuity, $f(x_{m_r}) \to f(l)$, $f(x_{n_r}) \to f(l')$. However, $f(x_{m_r}) = y_{m_r}$, $f(x_{n_r}) = y_{n_r}$ are both subsequences of (y_n), which converges to y. Hence $f(l) = f(l') = y$, which contradicts $l \neq l'$.

So (x_n) must converge. Suppose the limit is l. Then by continuity, $f(x_n) \to f(l)$. But $f(x_n) = y_n \to y$. Therefore $y = f(l)$, and hence $l = x$. □

4.27 Application: *n*th roots

For each positive integer $n \geq 2$, the function x^n (Fig. 4.15) is continuous and strictly increasing over $x \geq 0$. Therefore, by 4.26, there is a well-defined inverse function, which we shall denote by $y^{1/n}$ and call the *positive nth root* of y (Fig. 4.16). The domain of definition of $y^{1/n}$ is $y \geq 0$ and it is continuous for all these y.

For n even (Fig. 4.17), there is of course a negative nth root as well. For n odd (Fig. 4.18), there is only one (real) nth root, but in this case there is an nth root of negative numbers also. □

The last method we shall describe for generating continuous functions involves the use of infinite series. Consider the following example.

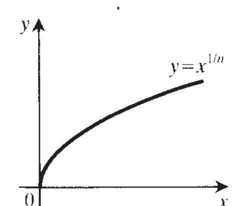


Fig. 4.16

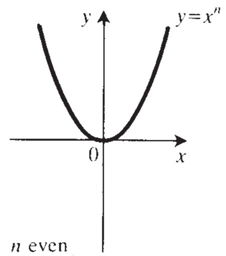


Fig. 4.17

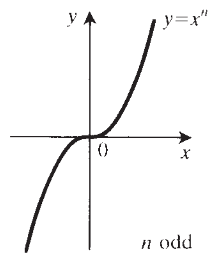


Fig. 4.18

4.28 Example

$$\sum_{0}^{\infty} x^n = \frac{1}{1-x} \qquad (|x|<1).$$

We can regard the series $\Sigma_0^\infty x^n$ as a series of *functions* whose nth term is the function x^n. The sum of the series is another function $1/(1-x)$. Observe that x^n is continuous for all $|x|<1$, and so is the sum function $1/(1-x)$. □

In general, however, we have no right to expect that the sum of an infinite series $\Sigma_1^\infty f_n(x)$ of continuous functions $f_n(x)$ is continuous. By induction on 4.19 (i), we can certainly say that the *partial* sums $\Sigma_1^N f_n(x)$ are continuous, but in the general situation we cannot say more than this. The following example gives an indication of the kind of thing that may happen (see Fig. 4.19).

4.29 Abel's example

$$\begin{aligned} \sum_{1}^{\infty} \frac{\sin nx}{n} &= \tfrac{1}{2}(\pi - x) \qquad (0<x<2\pi), \\ &= 0 \qquad (x=0, 2\pi). \end{aligned}$$ □

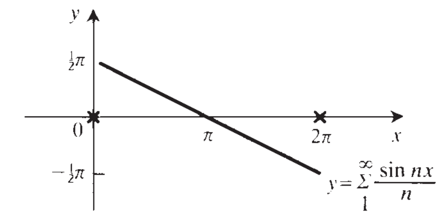


Fig. 4.19

We cannot give a proof of this result at this stage. (See Appendix, however.)

Observe that $(\sin nx)/n$ is continuous for all $x \in [0, 2\pi]$ but that the sum function has discontinuities at $x = 0, 2\pi$.

The historical importance of this example is that Abel published it in 1829 as a counter-example to an assertion by Cauchy that $\Sigma_1^\infty f_n(x)$ is always continuous if $f_n(x)$ is continuous for all n. In view of Abel's subsequent career, one cannot help wondering whether he should have been a little more tactful.

It turns out that we can say $\Sigma_1^\infty f_n(x)$ is continuous if we assume a little more about $f_n(x)$. In fact we have the following theorem.

4.30 Weierstrass' theorem

If $(f_n(x))_{n\geqslant 1}$ is a sequence of continuous functions on an interval $[a, b]$, and if there exists a convergent series $\Sigma_1^\infty M_n$ of constants M_n such that $|f_n(x)| \leqslant M_n$ for all $n \geqslant 1$ and for all $x \in [a, b]$, then the series $\Sigma_1^\infty f_n(x)$ converges for every $x \in [a, b]$, and its sum $s(x) = \Sigma_1^\infty f_n(x)$ is continuous on $[a, b]$. □

We shall call the condition that M_n with the above properties exists the *W-condition* (after Weierstrass). Observe that Abel's series $\Sigma_1^\infty (\sin nx)/n$ doesn't satisfy the W-condition over $[0, 2\pi]$.

Proof of 4.30 It is clear from the comparison test (3.13) that $\Sigma_1^\infty f_n(x)$ is absolutely convergent for each $x \in [a, b]$, so the sum function $s(x) = \Sigma_1^\infty f_n(x)$ is well defined over the interval $[a, b]$. We have to show it is continuous.

Suppose $c \in [a, b]$ and $\varepsilon > 0$ are given. Let N be chosen such that

$$\sum_{N+1}^{\infty} M_n < \tfrac{1}{3}\varepsilon$$

(see 3.38, question 16), and for each $n = 1, 2, \ldots, N$, let $\delta_n > 0$ be chosen such that

$$|f_n(x) - f_n(c)| < \frac{\varepsilon}{3N}$$

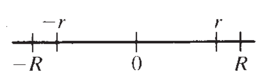
for all $|x - c| < \delta_n$. Then, if we set $\delta = \min_{1 \leqslant n \leqslant N} \delta_n$, we have $\delta > 0$ and, for any $|x - c| < \delta$,

$$\begin{aligned} |s(x) - s(c)| &= \left|\sum_1^\infty f_n(x) - \sum_1^\infty f_n(c)\right| \\ &= \left|\sum_1^N (f_n(x) - f_n(c)) + \sum_{N+1}^\infty f_n(x) - \sum_{N+1}^\infty f_n(c)\right| \\ &\leqslant \sum_1^N |f_n(x) - f_n(c)| + \sum_{N+1}^\infty |f_n(x)| + \sum_{N+1}^\infty |f_n(c)| \\ &< \sum_1^N \frac{\varepsilon}{3N} + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon \\ &= \varepsilon. \end{aligned}$$

Hence $s(x)$ is continuous at $x = c$ as required. □

4.31 Application

If the power series $\sum_0^\infty a_n x^n$ has radius of convergence R (see 3.33), then its sum is continuous for all $|x| < R$.



Proof We can show $\sum_0^\infty a_n x^n$ satisfies the W-condition on every closed interval $[-r, r]$ where $r < R$. In fact, for all $|x| \leqslant r$ we have

$$|a_n x^n| \leqslant |a_n|\, r^n,$$

and $\sum_0^\infty a_n r^n$ is absolutely convergent. (See proof of 3.33.) □

With the aid of 4.31 we can now substantially add to our stock of continuous functions. We shall use power series to give a rigorous definition of the exponential function e^x, which of course includes a definition of e as e^1. We shall define $\log_e x$ as the inverse function of e^x, and then use these two functions to define general powers and logarithms. We shall also give a rigorous definition of the trigonometric functions $\sin x$, $\cos x$ via power series, and obtain some of their basic properties, including their periodicity, and a definition of π in terms of solutions of the equation $\cos x = 0$.

4.32 Definition

We define the *exponential function* $e(x)$ to be

$$e(x) = \sum_0^\infty \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots \qquad (0! = 1).$$ □

The power series has radius of convergence $R = \infty$, so $e(x)$ is defined and continuous for all x by 4.31.

4.33 Theorem: Exponential theorem

$e(x + y) = e(x)e(y)$. □

Proof $e(x) = \sum_0^\infty x^n/n!$, $e(y) = \sum_0^\infty y^n/n!$ are both absolutely convergent, so, by 3.36, we have

$$\begin{aligned} e(x)e(y) &= \left(1 + x + \frac{x^2}{2!} + \cdots\right)\left(1 + y + \frac{y^2}{2!} + \cdots\right) \\ &= 1 + x + y + \frac{x^2}{2!} + xy + \frac{y^2}{2!} + \cdots \\ &= 1 + (x + y) + \frac{(x + y)^2}{2!} + \cdots \\ &= e(x + y). \end{aligned}$$

4.34 Definition

$e = e(1) = \sum_0^\infty (1/n!) = 1 + 1 + (1/2!) + \cdots$. □

It can be proved that $e = \lim_{n\to\infty} (1 + 1/n)^n$. (See 4.59, question 4.)

4.35 Theorem

For x rational, $e(x) = e^x$. □

Proof $e(0) = 1 = e^0$ is immediate from the definition of $e(x)$. $e(n) = (e(1))^n = e^n$ for all positive integers n follows from 4.33. Also

$$\left(e\left(\frac{1}{n}\right)\right)^n = e\left(\frac{n}{n}\right) = e(1) = e,$$

and clearly $e(1/n) > 0$, so $e(1/n) = e^{1/n}$ for all positive integers n. (See 4.27.)

If m, n are positive integers $\geqslant 2$, then

$$e\left(\frac{m}{n}\right) = \left(e\left(\frac{1}{n}\right)\right)^m = (e^{1/n})^m = e^{m/n}.$$

Finally, for any x we have

$$e(x)e(-x) = e(x-x) = e(0) = 1.$$

Therefore

$$e(-x) = \frac{1}{e(x)} = \frac{1}{e^x} = e^{-x}$$

if x is rational. □

4.36 Definition

For x irrational we define $e^x = e(x)$. □

Theorem 4.33 now takes on the more familiar form

$$e^{x+y} = e^x e^y.$$

4.37 Theorem

(i) e^x increases strictly for all x.
(ii) $e^x \to \infty$ as $x \to \infty$.
(iii) $e^x \to 0$ as $x \to -\infty$. □

Proof The fact that $e^x e^{-x} = e^{x-x} = e^0 = 1$ shows that $e^x \neq 0$ for any x. Therefore $e^x > 0$ for all x since, if e^x were ever negative, then, by the intermediate value theorem (4.24), it would have to vanish somewhere.

It is clear from the definition 4.32 that $e^x > 1$ for all $x > 0$. Therefore if $x < y$ we have

$$e^y = e^{y-x}e^x > e^x,$$

since $y - x > 0$ and $e^x > 0$. This shows e^x is strictly increasing.

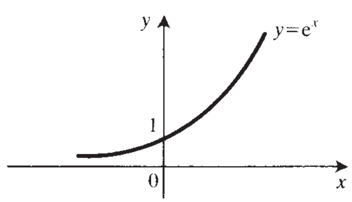


Fig. 4.20

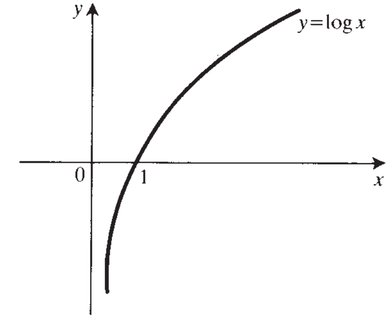


Fig. 4.21

To show $e^x \to \infty$ as $x \to \infty$, it is sufficient to observe that $e^x > x$ for all $x > 0$, another fact which is clear from 4.32.

It follows immediately that $e^x \to 0$ as $x \to -\infty$, since $e^x = 1/e^{-x}$ and $-x \to \infty$. □

4.38 Definition

For $x > 0$, we define the *logarithm* to the base e of x, denoted by $\log_e x$, or just $\log x$, to be the inverse function of e^x. □

The inverse function theorem (4.26) guarantees that $\log x$ is continuous for all $x > 0$. Clearly also $\log x$ is strictly increasing, tends to infinity as $x \to \infty$ and tends to minus infinity as $x \to 0_+$.

4.39 Theorem

$\log xy = \log x + \log y.$ □

Proof Let $X = \log x$, $Y = \log y$, $Z = \log xy$. Then we have $e^X = x$, $e^Y = y$, $e^Z = xy$. Therefore

$$e^Z = e^X e^Y = e^{X+Y}$$

and hence $Z = X + Y$ as required. □

4.40 Theorem

For x rational and $a > 0$

$$\log a^x = x \log a.$$ □

Proof 4.40 is clearly true for $x = 0, 1$.

For any positive integer $n \geqslant 2$ we have

$$\log a^n = n \log a$$

by 4.39, also

$$n \log a^{1/n} = \log a,$$

and so

$$\log a^{1/n} = \frac{1}{n} \log a.$$

Therefore, if m, n are positive integers,

$$\log a^{m/n} = m \log a^{1/n} = \frac{m}{n} \log a.$$

Finally, if x is a negative rational, then

$$\log a^x + \log a^{-x} = \log (a^x a^{-x}) = \log 1 = 0,$$

and so

$$\log a^x = -\log a^{-x} = x \log a. \qquad \square$$

4.41 Definition

For x irrational and $a > 0$

$$a^x = e^{x \log a}. \qquad \square$$

Observe that this formula is a *theorem* for x rational (4.40), but a *definition* for x irrational.

Theorem 4.23 shows a^x is a continuous function of x (and a). Theorem 4.37 shows a^x is strictly increasing if $a > 1$ (Fig. 4.22),

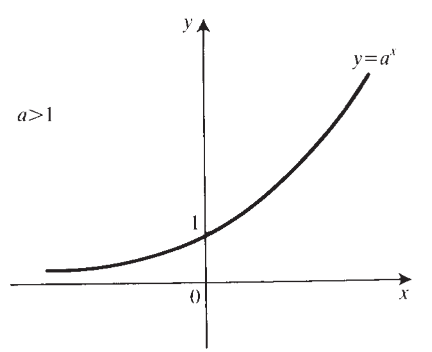


Fig. 4.22

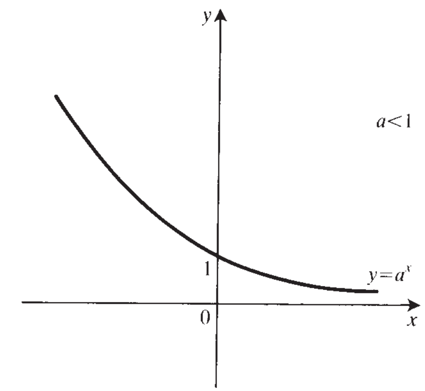


Fig. 4.23

strictly decreasing if $a < 1$ (Fig. 4.23), and takes on all positive values in both cases.

4.42 Exercise

Prove the laws of indices from the definition 4.41, i.e.,

(i) $a^{x+y} = a^x a^y$,
(i) $a^{xy} = (a^x)^y$,
(iii) $(ab)^x = a^x b^x$. $\square$

4.43 Definition

For $a > 0$, $x > 0$ we define $\log_a x$ to be the inverse function of a^x. $\square$

4.44 Exercise

Prove the laws of logarithms, i.e.,

(ii) $\log_a xy = \log_a x + \log_a y$,
(ii) $\log_a x^y = y \log_a x$,
(iii) $\log_a b \log_b c = \log_a c$. $\square$

So much for powers and logarithms. Now for the trigonometric functions.

As promised we define the sine and cosine functions by means of their Maclaurin series as follows.

4.45 Definition

For all x we define

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots. \qquad \square$$

For the time being we must pretend that we know nothing about $\sin x$, $\cos x$ and build up their properties from the definitions 4.45 as if we had never seen these functions before. Of course we know what to expect from geometrical considerations, but an *arithmetical* approach demands that we sham ignorance, and appear genuinely surprised when the arithmetic bears out the geometry.

The first thing to observe is that both defining series are power series with infinite radius of convergence, and so $\sin x$, $\cos x$ are both defined and continuous for all x by 4.31.

4.46 Theorem: Addition formulae

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y. \qquad \square$$

Proof The series for sine and cosine are both absolutely convergent for all values of the variable, and so it is legitimate to multiply them together by 3.36. Therefore we have

$$\begin{aligned}
&\sin x \cos y + \cos x \sin y \\
&= \left(x - \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{y^2}{2!} + \cdots\right) + \left(1 - \frac{x^2}{2!} + \cdots\right)\left(y - \frac{y^3}{3!} + \cdots\right) \\
&= x + y - \frac{x^3}{3!} - \frac{xy^2}{2!} - \frac{x^2y}{2!} - \frac{y^3}{3!} + \cdots \\
&= (x+y) - \frac{(x+y)^3}{3!} + \cdots \\
&= \sin(x+y).
\end{aligned}$$

The second formula is proved similarly. $\square$

4.47 Theorem

$\sin^2 x + \cos^2 x = 1.$ $\square$

Proof Put $y = -x$ in the cosine formula 4.46 and use the facts that $\cos 0 = 1$, $\cos(-x) = \cos x$, $\sin(-x) = -\sin x$, all easily provable from the definition 4.45. $\square$

4.48 Corollary

$|\sin x| \le 1$, $|\cos x| \le 1$. $\square$

4.49 Theorem

$\sin x > 0$ for all $0 < x < 2$. $\square$

Proof We have

$$\begin{aligned}
\sin x &= \left(x - \frac{x^3}{3!}\right) + \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right) + \cdots \\
&= \frac{x}{3!}(6 - x^2) + \frac{x^5}{7!}(42 - x^2) + \cdots
\end{aligned}$$

which is clearly positive for all $0 < x < 2$.

4.50 Theorem

$\cos x$ strictly decreases for all $0 \le x \le 2$. $\square$

Proof If $0 \le x < y \le 2$, then

$$\cos x - \cos y = 2 \sin\frac{x+y}{2} \sin\frac{y-x}{2}$$

(easily deducible from 4.46) which is positive by 4.49. $\square$

4.51 Theorem

$\cos 2 < 0.$ $\square$

Proof We have

$$\begin{aligned}
\cos 2 &= 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} + \frac{2^8}{8!} - \cdots \\
&= 1 - 2 + \frac{2}{3} - \frac{2^6}{8!}(56 - 4) - \cdots
\end{aligned}$$

which is clearly negative. $\square$

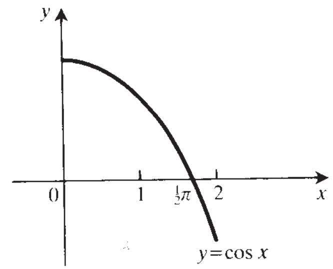


Fig. 4.24

4.52 Corollary

$\cos x$ vanishes precisely once in the interval $0 < x < 2$. □

Proof We have $\cos 0 = 1$ from the definition, so the result follows by the intermediate value theorem and the fact that $\cos x$ decreases strictly over $[0, 2]$. □

4.53 Definition of π

We define π to be the unique real number such that $0 < \frac{1}{2}\pi < 2$ and $\cos \frac{1}{2}\pi = 0$ (see Fig. 4.24). □

4.54 Theorem

$\sin \frac{1}{2}\pi = 1.$ □

Proof Follows from 4.47 and 4.49. □

4.55 Corollaries

$$\sin\left(x + \frac{\pi}{2}\right) = \cos x, \qquad \cos\left(x + \frac{\pi}{2}\right) = -\sin x.$$ □

Proof Follows from 4.46, 4.53 and 4.54. □

4.55 enables the full behaviour of $\sin x$, $\cos x$ to be built up from the behaviour of $\cos x$ on $[0, \frac{1}{2}\pi]$. The graphs are as shown in Figs 4.25 and 4.26.

Observe that $\sin x$, $\cos x$ are both *periodic* with *period* 2π in the

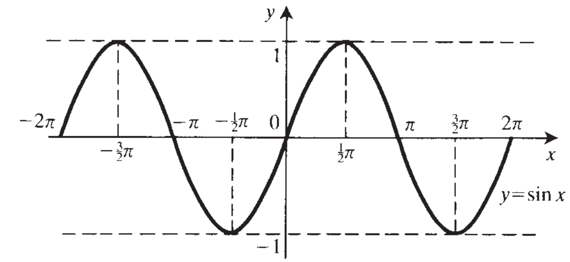


Fig. 4.25

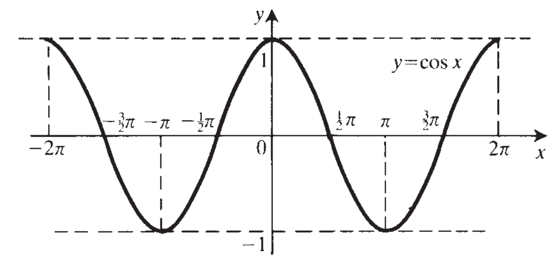


Fig. 4.26

sense that they repeat themselves every 2π, i.e.,

$$\sin(x + 2\pi) = \sin x, \qquad \cos(x + 2\pi) = \cos x.$$

Rigorous definitions of the other trigonometric functions can now be given by defining them in terms of sine and cosine. In fact, we have the following.

4.56 Definitions

(i) $\tan x = \dfrac{\sin x}{\cos x} \qquad (x \neq (n + \frac{1}{2})\pi)$

(ii) $\cot x = \dfrac{\cos x}{\sin x} \qquad (x \neq n\pi)$

(iii) $\sec x = \dfrac{1}{\cos x} \qquad (x \neq (n + \frac{1}{2})\pi)$

(iv) $\operatorname{cosec} x = \dfrac{1}{\sin x} \qquad (x \neq n\pi)$

We now have a sufficient supply of continuous functions, and know almost enough of their basic properties to be able to commence our treatment of the calculus. One property still needs to be considered, and that is the so-called 'minimax' property of the next theorem.

4.57 Definition

We say $f(x)$ is *bounded above* on an interval I if there exists M, called an *upper bound,* such that $f(x) \leq M$ for all $x \in I$. □

For example, $f(x) = \sin x$ is bounded above on $(0, \pi)$, but $f(x) = 1/x$ isn't.

We define bounded *below* and *lower* bound similarly, and say $f(x)$ is *bounded* if bounded above and below.

By the upper bound axiom, any $f(x)$ bounded above on an interval I must have a supremum, denoted by $\sup_{x \in I} f(x)$. This is a *maximum,* denoted by $\max_{x \in I} f(x)$, if and only if it is attained on I.

For example, $\sup_{0<x<\pi} \cos x = 1$, but there is no $x \in (0, \pi)$ such that $\cos x = 1$. On the other hand, $\max_{0<x<\pi} \sin x = 1$ exists since $\sin x = 1$ at $x = \frac{1}{2}\pi \in (0, \pi)$.

Similar remarks apply to $\inf_{x \in I} f(x)$ and $\min_{x \in I} f(x)$.

The minimax theorem says that under certain circumstances it is possible to assume that continuous functions always have maxima and minima.

4.58 Minimax theorem

If $f(x)$ is continuous on the bounded closed interval $[a, b]$, then $f(x)$ is bounded on $[a, b]$ and attains its supremum and infimum on $[a, b]$ (see Fig. 4.27). □

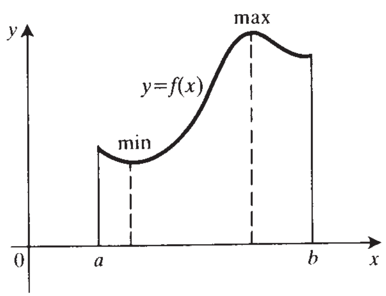


Fig. 4.27

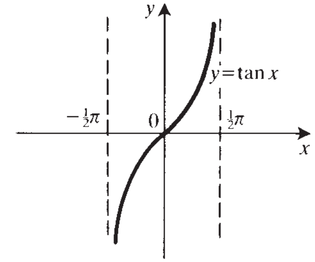


Fig. 4.28

Proof Assume (for contradiction) that $f(x)$ is unbounded above. Then (see 2.39, question 9) we can find a sequence (x_n) in $[a, b]$ such that $f(x_n)$ diverges to infinity. However, by the Bolzano–Weierstrass theorem (2.38), (x_n) must have a convergent subsequence $(x_{n_r})_{r\geq 1}$. Suppose $(x_{n_r})_{r\geq 1}$ converges to l. Then clearly $l \in [a, b]$ so, by continuity, $f(x_{n_r})$ must converge to $f(l)$. But $f(x_{n_r})$, being a subsequence of $f(x_n)$, must diverge to infinity (see 2.37). This is the required contradiction.

Let $M = \sup_{a \leq x \leq b} f(x)$. We want to show that there must exist $x \in [a, b]$ such that $f(x) = M$. We can certainly find a sequence (x_n) in $[a, b]$ such that $f(x_n) \to M$. (See 2.39, question 8.) By Bolzano–Weierstrass, there is a convergent subsequence $(x_{n_r})_{r\geq 1}$ with limit l say. Again, we clearly have $l \in [a, b]$ and therefore, by continuity, $f(x_{n_r}) \to f(l)$. But $f(x_{n_r})$is a subsequence of $f(x_n)$ which tends to M. Hence $f(l) = M$ as required.

The proofs that $f(x)$ is bounded below and attains its infimum on $[a, b]$ are similar. □

We should observe that the minimax theorem is no longer true if the interval $[a, b]$ is replaced by an unbounded or an unclosed interval. For example, $\tan x$ is continuous on the open interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, but is unbounded above and below (see Fig. 4.28).

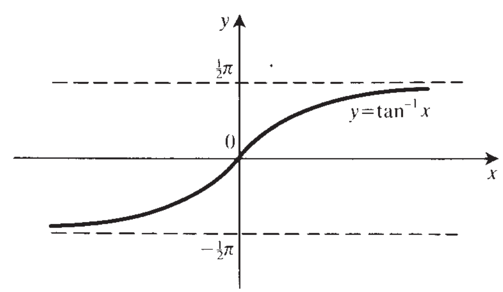


Fig. 4.29

Also $\tan^{-1} x$ is continuous for all x, taken as the inverse function of $\tan x$ on the interval $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, and, though it is bounded over all x, doesn't attain its supremum or infimum (see Fig. 4.29).

4.59 Miscellaneous exercises

1. Discuss the continuity of the following functions.
(i) $\operatorname{sgn} \sin x$ (ii) $\cos \operatorname{sgn} x$ (iii) $x \operatorname{sgn} x$

$$\text{(iv) } f(x) = \operatorname{sgn} \sin \frac{1}{x} \quad (x \neq 0),$$
$$= 0 \quad (x = 0).$$
$$\text{(v) } f(x) = x \sin \frac{1}{x} \quad (x \neq 0),$$
$$= 0 \quad (x = 0).$$
$$\text{(vi) } f(x) = 1 \quad \text{if } x \text{ rational},$$
$$= 0 \quad \text{if } x \text{ irrational}.$$
$$\text{(vii) } f(x) = x \quad \text{if } x \text{ rational},$$
$$= 1 - x \quad \text{if } x \text{ irrational}.$$

Draw graphs (where possible).
2. Evaluate the following limits.

$$\text{(i) } \lim_{x\to 0} \frac{e^x - 1}{x} \qquad \text{(ii) } \lim_{x\to 0} \frac{\sin x}{x} \qquad \text{(iii) } \lim_{x\to 0} \frac{1 - \cos x}{x^2}$$

Hint: Use 4.31.
3. Prove that

$$1 + x \leqslant e^x \leqslant \frac{1}{1-x}$$

for all $|x| < 1$. *Hint:* Observe that $\sum_0^\infty x^n/n! \leqslant \sum_0^\infty x^n$ $(0 \leqslant x < 1)$.
4. Prove that

$$\frac{x}{1+x} \leqslant \log(1+x) \leqslant x$$

for all $|x| < 1$. *Hint:* Take logarithms in the inequalities of question 3.
Deduce that

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1.$$

Hence prove

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Hint: Use 4.41, 4.3 and the continuity of e^x.
5. Prove that, for any positive integer n,

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \infty.$$

Hint: Compare $\sum_0^\infty x^n/n!$ with its $(n+1)$th term.
Deduce that

(i) $\lim_{x\to\infty} x^4 e^{-x} = 0$,

(ii) $\lim_{n\to\infty} n^2 e^{-\sqrt{n}} = 0$ (see remark after 4.16)

(iii) $\sum_1^\infty e^{-\sqrt{n}}$ converges.

Observe: Exponentials dominate powers.
6. Prove that

$$\text{(i) } \lim_{x\to\infty} \frac{\log x}{x} = 0, \qquad \text{(ii) } \lim_{x\to 0} x \log x = 0.$$

Hint: Put $x = e^t$ and let $t \to \pm\infty$. *Observe:* Powers dominate logarithms.
Discuss the convergence of $\sum_1^\infty (-1)^{n-1}(\log n)/n$.
7. Prove that

$$\left|1 - \frac{\sin x}{x}\right| < \frac{x^2}{1-x^2}$$

for all $0 < |x| < 1$.
8. Prove that if $f(x)$ is continuous on $[0, 1]$ and $f(0) = 1$, $f(1) = 0$, then $f(x) = x$ for some $x \in (0, 1)$. *Hint:* Apply the intermediate value theorem to $g(x) = f(x) - x$.
9. Prove that if $f(x)$ is continuous on $[0, 1]$ and $f(x) \neq 0$ for all $x \in [0, 1]$, then $1/f(x)$ is bounded on $[0, 1]$. *Hint:* Use the minimax theorem to show $\inf_{0\leqslant x\leqslant 1} |f(x)|$ is positive.
10. Prove that, if $f(x)$ is continuous on $[a, b]$, and if $\varepsilon > 0$ is given, then there exists a smallest $x \in [a, b]$ such that $|f(x) - f(a)| = \varepsilon$, unless $|f(x) - f(a)| < \varepsilon$ for all $x \in [a, b]$. *Hint:* Consider $\inf E$, where

$$E = \{x \in [a, b] : |f(x) - f(a)| = \varepsilon\}.$$

The intermediate value theorem shows E is not empty.