# 3
# Infinite series

In common parlance the words 'sequence' and 'series' mean much the same thing. In mathematics, however, they have different meanings. Whereas a *sequence* is a succession of numbers,

$$a_1, a_2, a_3, \ldots, a_n, \ldots,$$

a *series* is a succession of numbers which are supposed to be added together,

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots.$$

An *infinite series* simply means an *infinite sum.*

To get a precise definition of an infinite sum, we have to regard it as a limit of finite sums. If we write

$$s_n = a_1 + a_2 + \cdots + a_n$$

for each $n$, and if the *sequence* $(s_n)_{n\geqslant 1}$ converges to a limit $s$, then we say the infinite sum

$$a_1 + a_2 + \cdots + a_n + \cdots$$

has the value $s$.

For example, suppose $a_n = 1/2^n$. Then

$$s_n = \tfrac{1}{2} + \tfrac{1}{4} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

which converges to 1. So we write

$$\tfrac{1}{2} + \tfrac{1}{4} + \cdots + \frac{1}{2^n} + \cdots = 1.$$

Infinite series occur at several strategic points in the development of analysis. For instance, Taylor's theorem in the differential calculus says that, under suitable conditions, any function $f(x)$ can be expanded as

$$f(x) = f(a) + (x-a)f'(a) + \tfrac{1}{2}(x-a)^2 f''(a) + \cdots,$$

where $f'(a)$, $f''(a)$ are successive derivatives of $f(x)$ evaluated at



$x = a$. Important cases are

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots,$$

all of which are actually called *Maclaurin* expansions by a quirk of language. Not only are these expansions useful for approximating the functions in question, they also provide us with a completely rigorous method of *defining* these functions which is not dependent on geometrical intuition. (See Chapter 4.)

It is therefore relevant to make a rigorous analysis of infinite series at an early stage, and the weapons for achieving this are ready to hand in the form of the theory of convergence of infinite sequences as developed in Chapter 2.

Before making the necessary definitions and commencing the theory, it will be useful to set up the summation notation. We shall use the following abbreviations.

$$\sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N.$$

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots.$$

We call $\sum_{n=1}^{N} a_n$ the *Nth partial* sum of the *infinite series* $\sum_{n=1}^{\infty} a_n$. We call $a_n$ the *nth term.* We shall often trim down the notation to $\sum_1^N a_n$, $\sum_1^\infty a_n$, and write $s_N = \sum_1^N a_n$, $s = \sum_1^\infty a_n$.

## 3.1 Definition

We say the infinite series $\sum_1^\infty a_n$ *converges* (or is *convergent*) if the infinite sequence $(s_N)_{N\geqslant 1}$ of $N$th partial sums converges. We call the limit $s$ of $(s_N)_{N\geqslant 1}$, if it exists, the *sum* of the series $\sum_1^\infty a_n$, and also denote it by $\sum_1^\infty a_n$. □

For example, $\sum_1^\infty 1/2^n$ converges and its sum is 1, and we also write $\sum_1^\infty 1/2^n = 1$.

In other cases we describe the behaviour of $\sum_1^\infty a_n$ by saying what the behaviour of $(s_N)_{N\geqslant 1}$ is, e.g. $\sum_1^\infty a_n$ diverges to infinity, oscillates, etc.

## 3.2 Example: Geometric series

$\sum_1^\infty x^n$ where $x$ is fixed.

Here we have

$$s_N = x + x^2 + \cdots + x^N.$$

Therefore

$$xs_N = x^2 + x^3 + \cdots + x^{N+1}.$$

Subtracting, we obtain

$$(1-x)s_N = x - x^{N+1},$$

which gives

$$s_N = \frac{x - x^{N+1}}{1-x},$$

provided $x \neq 1$. Clearly $s_N = N$ if $x = 1$.

It follows that $(s_N)$ converges if $|x| < 1$, and diverges if $|x| \geqslant 1$ (see 2.18 and 2.26). In fact,

$$\sum_1^\infty x^n = \frac{x}{1-x}$$

if $|x| < 1$, whereas $\sum_1^\infty x^n$ diverges to infinity if $x \geqslant 1$ and oscillates if $x \leqslant -1$. □

## 3.3 Exercises

(i) Show that the $N$th partial sum of the series $\sum_1^\infty 1/n(n+1)$ is

$$\sum_1^N \frac{1}{n(n+1)} = \frac{N}{N+1}.$$

*Hint*: Use induction, or resolve $1/n(n+1)$ into partial fractions.

(ii) Deduce that the series $\sum_1^\infty 1/n(n+1)$ converges, and find its sum. □

## 3.4 Example: The harmonic series

$\sum_1^\infty 1/n$. Here we have

$$s_N = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{N}$$

for which there is no simple formula which enables the convergence or otherwise of $(s_N)$ to be quickly decided. However, we can say

$$\begin{aligned} s_{2^N} &= 1 + \tfrac{1}{2} + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}) + \cdots + \left(\frac{1}{2^{N-1}+1} + \cdots + \frac{1}{2^N}\right) \\ &> 1 + \tfrac{1}{2} + (\tfrac{1}{4} + \tfrac{1}{4}) + (\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}) + \cdots + \left(\frac{1}{2^N} + \cdots + \frac{1}{2^N}\right) \\ &= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \cdots + \tfrac{1}{2} \\ &= 1 + \tfrac{1}{2}N. \end{aligned}$$

Therefore, by the open sandwich principle (2.25), it follows that $(s_{2^N})_{N\geqslant 1}$ diverges to infinity. Now, $(s_N)_{N\geqslant 1}$ is monotonic increasing, so must either converge or diverge to infinity (see 2.33). If $(s_N)_{N\geqslant 1}$ were to converge, then $(s_{2^N})_{N\geqslant 1}$, being a subsequence, would also converge (see 2.37). But we have just shown $(s_{2^N})_{N\geqslant 1}$ diverges to infinity. Hence $(s_N)_{N\geqslant 1}$ must diverge to infinity, i.e., $\sum_1^\infty 1/n$ diverges to infinity. □



## 3.5 Exercise

Show by a similar method that the series $\sum_1^\infty 1/\sqrt{n}$ diverges. □

## 3.6 Example: Euler's series

$\sum_1^\infty 1/n^2$. This time we have

$$s_N = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{N^2}$$

for which there is again no simple formula. However, the method of 3.4 can again be applied, but now with the inequalities reversed, to give

$$\begin{aligned} s_{2^N} &= 1 + \frac{1}{2^2} + \left(\frac{1}{3^2} + \frac{1}{4^2}\right) + \left(\frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2}\right) \\ &\quad + \cdots + \left(\frac{1}{(2^{N-1}+1)^2} + \cdots + \frac{1}{(2^N)^2}\right) \\ &< 1 + \frac{1}{2^2} + \left(\frac{1}{2^2} + \frac{1}{2^2}\right) + \left(\frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2}\right) \\ &\quad + \cdots + \left(\frac{1}{(2^{N-1})^2} + \cdots + \frac{1}{(2^{N-1})^2}\right) \\ &= 1 + \tfrac{1}{4} + \tfrac{1}{2} + \tfrac{1}{4} + \cdots + \frac{1}{2^{N-1}} \\ &< \tfrac{9}{4} \end{aligned}$$

for all $N$. So $(s_{2^N})_{N\geq 1}$ is increasing bounded above, therefore converges. But therefore, also, the full sequence $(s_N)_{N\geq 1}$ must converge, since otherwise it would diverge to infinity, which would involve its subsequence $(s_{2^N})_{N\geq 1}$ also diverging to infinity. Hence $\sum_1^\infty 1/n^2$ converges. □

Observe that, even though we can show $\sum_1^\infty 1/n^2$ converges by the above method, we cannot say what its sum is. In fact, the sum is known to be $\pi^2/6$, a fact which was first discovered by Euler in 1734, but a proof of this result is beyond the scope of this book. (See Appendix.)

## 3.7 Exercise

Show that the series $\sum_1^\infty 1/n^3$ converges by the method of 3.6. □

The problem with series in general is that we cannot expect a simple formula for the $N$th partial sum. This means that indirect methods have to be used to determine whether or not a series converges (as in 3.4 and 3.6). Such methods may give little or no information as to the actual value of the sum of a convergent series. For many purposes, however, it will be enough to know that a series does converge, without worrying too much about what its sum is.

In order to enable quick decisions to be made on convergence of series, a number of simple tests have been devised. These mainly involve reference to the $n$th term rather than the $N$th partial sum, clearly a simplifying manoeuvre. The tests we shall describe, whilst not forming an exhaustive collection, should nevertheless enable the reader to cope with most series he is likely to encounter in real life.

## 3.8 Theorem

(i) If $\sum_1^\infty a_n$, $\sum_1^\infty b_n$ both converge, then so does $\sum_1^\infty (a_n + b_n)$, and

$$\sum_1^\infty (a_n + b_n) = \sum_1^\infty a_n + \sum_1^\infty b_n.$$

(ii) If $\sum_1^\infty a_n$ converges, and $C$ is constant, then $\sum_1^\infty (Ca_n)$ converges, and its sum is

$$\sum_1^\infty (Ca_n) = C\sum_1^\infty a_n.$$



**Proof** (i) For all $N$ we have

$$\sum_1^N (a_n + b_n) = \sum_1^N a_n + \sum_1^N b_n,$$

so, taking limits and applying 2.11, we deduce immediately that $\sum_1^\infty (a_n + b_n)$ is convergent, and its sum is given by

$$\sum_1^\infty (a_n + b_n) = \sum_1^\infty a_n + \sum_1^\infty b_n.$$

(ii) Follows similarly by taking limits in the formula

$$\sum_1^N (Ca_n) = C\sum_1^N a_n.$$

□

## 3.9 Applications

(i) $\displaystyle\sum_1^\infty \left(\frac{1}{2^n} + \frac{1}{n^2}\right) = 1 + \frac{\pi^2}{6}.$

(ii) The general 'geometric series'

$$a + ar + ar^2 + \cdots + ar^n + \cdots$$

having first term $a$, and common ratio $r$, converges if $|r| < 1$, and its sum in this case is $a/(1-r)$.

In fact, by 3.2, we have

$$\begin{aligned} a + ar + ar^2 + \cdots &= a + a\sum_1^\infty r^n \\ &= a + \frac{ar}{1-r} \\ &= \frac{a}{1-r} \end{aligned}$$

if $|r| < 1$. □

## 3.10 Theorem

If $\sum_1^\infty a_n$ converges, then the sequence $(a_n)_{n\geq 1}$ of $n$th terms is null.

**Proof** Let $s_N = \sum_1^N a_n$, and suppose that $(s_N)_{N\geq 1}$ converges to $s$. Then we have

$$a_n = s_n - s_{n-1}$$

and so $(a_n)_{n\geq 1}$ must converge to $s - s = 0$. □

## 3.11 Corollary: Non-null test

If $(a_n)$ is *not* null, then $\sum_1^\infty a_n$ must diverge. □

## 3.12 Application

The non-null test can be used to give another proof of the divergence of $\sum_1^\infty x^n$ when $|x| \geqslant 1$. (See 3.2.) In fact, $(x^n)_{n\geqslant 1}$ is not null for $|x| \geqslant 1$ (see 2.26), so the result is immediate. □

Please note that the non-null test is a test for *divergence* only. One cannot prove a series $\sum_1^\infty a_n$ *converges* by demonstrating that its sequence of $n$th terms $(a_n)_{n\geqslant 1}$ is null. For example, $(1/n)_{n\geqslant 1}$ is null and yet $\sum_1^\infty 1/n$ diverges (see 3.4).

Faced with a series whose $n$th term is null, one has to try some other test, such as the following.

## 3.13 Comparison test

If $0 \leqslant a_n \leqslant b_n$ for all $n$, and if $\sum_1^\infty b_n$ converges, then also $\sum_1^\infty a_n$ converges.

**Proof** If we write $s_N = \sum_1^N a_n$, $t_N = \sum_1^N b_n$, then we have $s_N \leqslant t_N$ for all $N$, and we know $(t_N)_{N\geqslant 1}$ converges. It follows (by 2.29) that $(t_N)$ is bounded above, and therefore so is $(s_N)$. However, $(s_N)$ is *increasing*, on account of the fact that $a_n \geqslant 0$ for all $n$. Hence $(s_N)$ must converge (by 2.33), i.e., $\sum_1^\infty a_n$ converges. □

Observe that the condition $a_n \geqslant 0$ for all $n$ is crucial. Without this condition we could not say $(s_N)$ is monotonic, and the proof of 3.13 would fail.

## 3.14 Example

$$\sum_1^\infty \frac{n+5}{n^3+7} \text{ converges.}$$

In fact, we have

$$0 \leqslant \frac{n+5}{n^3+7} \leqslant \frac{6n}{n^3} = \frac{6}{n^2}$$

for all $n \geqslant 1$, and $\sum_1^\infty (6/n^2)$ converges by 3.4 and 3.8. □



## 3.15 Exercise

Show the following series converge.

$$\text{(i)} \sum_1^\infty \frac{n^2+2}{n^4+7} \qquad \text{(ii)} \sum_1^\infty \frac{2^n+3^n}{4^n+5^n}.$$ □

## 3.16 Corollary to 3.13

If $0 \leqslant a_n \leqslant b_n$ for all $n$, and if $\sum_1^\infty a_n$ diverges, then also $\sum_1^\infty b_n$ diverges.

**Proof** Follows immediately from 3.13. □

## 3.17 Example

$$\sum_1^\infty \frac{n+5}{n^2+7} \text{ diverges.}$$

In fact, we have

$$\frac{n+5}{n^2+7} \geqslant \frac{n}{8n^2} = \frac{1}{8n}$$

for all $n \geqslant 1$, and $\sum_1^\infty (1/8n)$ diverges (by 3.2 and 3.8). □

## 3.18 Exercises

Show the following series diverge.

$$\text{(i)} \sum_1^\infty \frac{n^2+2}{n^3+7} \qquad \text{(ii)} \sum_1^\infty \frac{6^n+5^n}{4^n+3^n}.$$ □

The following slightly more general form of the comparison test is sometimes useful.

## 3.19 Theorem

If $0 \leqslant a_n \leqslant b_n$ for all $n \geqslant$ some $N$, and if $\sum_1^\infty b_n$ converges, then so does $\sum_1^\infty a_n$.

**Proof** If $s_n = \sum_{r=1}^n a_r$, $t_n = \sum_{r=1}^n b_r$, then we have

$$s_n - s_N = \sum_{r=N+1}^n a_r \leqslant \sum_{r=N+1}^n b_r = t_n - t_N$$

for all $n > N$. Hence, as in the proof of 3.13, the boundedness of $(t_n)$ implies the boundedness of $(s_n)$, and $(s_n)$ increases for $n \geq N$, so converges by 2.33. □

## 3.20 Example

$$\sum_1^\infty \left(\frac{1}{2}+\frac{1}{n}\right)^n \text{ converges.}$$

In fact, we have

$$\left(\frac{1}{2}+\frac{1}{n}\right)^n \leq \left(\frac{5}{6}\right)^n$$

for all $n \geq 3$, so the result follows by 3.2 and 3.19. □

Our next test is one of the easiest tests to apply in that it gives a direct method of determining convergence or divergence of many series by evaluation of a simple limit.

## 3.21 Ratio test

If $a_n > 0$ for all $n \geq 1$, and if the sequence $(a_{n+1}/a_n)_{n\geq 1}$ converges to a limit $l$, then the series $\sum_1^\infty a_n$ converges if $l < 1$, and diverges if $l > 1$.

**Proof**

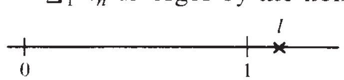
**Case l < 1** Let $r = \frac{1}{2}(l+1)$, so that $l < r < 1$. Then there must exist $N$ such that


$a_{n+1}/a_n < r$ for all $n \geq N$. (Play $\varepsilon = r - l$.) Therefore, if $n > N$, we have

$$0 < a_n = \frac{a_n}{a_{n-1}}\frac{a_{n-1}}{a_{n-2}}\cdots\frac{a_{N+1}}{a_N}a_N < r^{n-N}a_N,$$

and $\sum_{n=1}^\infty r^{n-N}a_N$ converges, since it is a geometric series with common ratio $r$ satisfying $0 < r < 1$. Hence $\sum_1^\infty a_n$ converges by 3.19.

**Case l > 1** In this case there exists $N$ such that $a_{n+1}/a_n > 1$ for all $n \geq N$. (Play $\varepsilon = l - 1$.) Therefore $(a_n)$ increases (strictly) for $n \geq N$, and so cannot be null. Hence $\sum_1^\infty a_n$ diverges by the non-null test (3.11). □






Observe that we say nothing about the case $l = 1$. In fact, if $l = 1$, $\sum_1^\infty a_n$ may either converge or diverge. For example, $\sum_1^\infty 1/n^2$ converges and $\sum_1^\infty 1/n$ diverges, and yet $l = 1$ in both cases.

## 3.22 Example

$\sum_1^\infty n/2^n$ converges.

In fact, writing $a_n = n/2^n$, we have

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{2n}$$

which converges to $\frac{1}{2}$, so we are in the case $l < 1$. □

## 3.23 Exercises

Use the ratio test to discover which of the following series converge:

$$\text{(i)} \sum_1^\infty \frac{n^2}{2^n} \qquad \text{(ii)} \sum_1^\infty \frac{n!}{2^n}$$

□

The condition $a_n \geq 0$ for the comparison test and tests based on it, such as the ratio test (see also 3.38, questions 7, 8, 9), may seem unduly restrictive. In fact, these tests are still useful for series $\sum_1^\infty a_n$ which don't satisfy this condition. This is because we can always test the corresponding series $\sum_1^\infty |a_n|$, which does satisfy the condition, and then use the following theorem.

## 3.24 Theorem

If $\sum_1^\infty |a_n|$ converges, then so does $\sum_1^\infty a_n$. □

## 3.25 Definition

We say the series $\sum_1^\infty a_n$ converges *absolutely* (or is *absolutely* convergent) if the series of absolute values $\sum_1^\infty |a_n|$ converges.

For example, the alternating series

$$\sum_1^\infty \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

converges absolutely, whilst

$$\sum_1^\infty \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

doesn't.

3.24 says absolute convergence implies convergence. So, e.g. $\Sigma_1^\infty(-1)^{n-1}/n^2$ converges. Of course, 3.24 says nothing about the convergence of $\Sigma_1^\infty(-1)^{n-1}/n$, since this series is not absolutely convergent. In fact, we shall see shortly that $\Sigma_1^\infty(-1)^{n-1}/n$ *is* convergent, which shows that the converse of 3.24 is false.

**Proof of 3.24** Let $a_n^+$, $a_n^-$ be defined as follows.

$$\begin{aligned} a_n^+ &= a_n && \text{if } a_n \geqslant 0,\\ &= 0 && \text{otherwise.}\\ a_n^- &= |a_n| && \text{if } a_n \leqslant 0,\\ &= 0 && \text{otherwise.}\end{aligned}$$

For example, if

$$\sum_1^\infty a_n = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots$$

then

$$\sum_1^\infty a_n^+ = 1 + 0 + \tfrac{1}{3} + 0 + \tfrac{1}{5} + 0 + \cdots,$$

$$\sum_1^\infty a_n^- = 0 + \tfrac{1}{2} + 0 + \tfrac{1}{4} + 0 + \tfrac{1}{6} + \cdots.$$

Observe that $a_n^+ \geqslant 0$, $a_n^- \geqslant 0$, and

(i) $a_n^+ + a_n^- = |a_n|$,

(ii) $a_n^+ - a_n^- = a_n$.

It follows from (i) and the comparison test that $\Sigma_1^\infty a_n^+$, $\Sigma_1^\infty a_n^-$ both converge. Therefore, by (ii) and 3.8, $\Sigma_1^\infty a_n$ converges. □

For series which are not absolutely convergent, the question of convergence remains unresolved, and is in many cases not easy to resolve. However, there is one class of series for which convergence can be established without undue difficulty by means of the following test.

## 3.26 Alternating series test

If $(a_n)_{n\geqslant 1}$ is a decreasing null sequence, then the alternating series $\Sigma_1^\infty(-1)^{n-1}a_n$ converges. □



## 3.27 Example

$\Sigma_1^\infty(-1)^{n-1}/n$ converges.

In fact, $a_n = 1/n$ clearly satisfies the conditions of 3.26. □

Hence, as claimed earlier, Theorem 3.24 has no converse. We emphasize this fact by making the following definition.

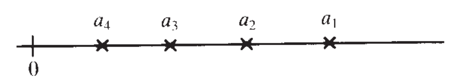
## 3.28 Definition

The series $\Sigma_1^\infty a_n$ will be said to converge *conditionally* (or to be *conditionally* convergent) if it converges, but not absolutely. □

So, e.g., $\Sigma_1^\infty(-1)^{n-1}/n$ converges conditionally.

**Proof of 3.26** If we write $s_N = \Sigma_1^N(-1)^{n-1}a_n$, then we have

$$\begin{aligned} s_{2N} &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2N-1} - a_{2N})\\ &\leqslant a_1,\end{aligned}$$




showing that $(s_{2N})_{N\geqslant 1}$ is increasing bounded above, therefore convergent to $s$, say. But then

$$s_{2N-1} = s_{2N} + a_{2N}$$

also converges to $s$. Hence the full sequence $(s_N)_{N\geqslant 1}$ converges to $s$.

## 3.29 Exercises

Which of the following series converge absolutely or conditionally?

$$\text{(i)}\ \sum_1^\infty(-1)^n\frac{n+1}{n^2+1} \qquad \text{(ii)}\ \sum_1^\infty(-1)^n\frac{n+1}{n^3+1}$$

N.B. Don't forget to check that $a_n$ *decreases* when applying the alternating series test. □

We shall now illustrate the application of the tests so far mentioned by considering a certain class of series known as 'power series' defined as follows.

## 3.30 Definition

A *power series* is a series of the form $\sum_{n=0}^{\infty} a_n x^n$, where $(a_n)_{n\geq 0}$ is a sequence of constants, and $x$ is a variable. □

Our interest in power series stems from the fact that Maclaurin series are power series, and that it is our intention to use such series to define some of the important functions of analysis in the next chapter.

## 3.31 Examples

(i) $\sum_0^\infty x^n$. This is the case $a_n = 1$ for all $n$. This power series is already known to be absolutely convergent if $|x| < 1$, and divergent otherwise. (See 3.2.)

(ii) $\sum_1^\infty x^n/n$. If we write $b_n = x^n/n$, then we have

$$\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{x^{n+1}}{n+1}\frac{n}{x^n}\right| = \frac{n\,|x|}{n+1},$$

which converges to $|x|$. Therefore, by the ratio test (3.21), this power series is absolutely convergent for $|x| < 1$, and divergent if $|x| > 1$ (since its sequence of $n$th terms is then not null, see proof of 3.21).

This leaves the cases $x = \pm 1$. If $x = 1$, the series is $\sum_1^\infty 1/n$, which we know diverges. If $x = -1$, the series is $\sum_1^\infty (-1)^n/n$, which converges by 3.27 and 3.8 (ii).

Hence $\sum_1^\infty x^n/n$ converges if $-1 \leq x < 1$, and diverges otherwise.

(iii) $\sum_1^\infty x^n/n^2$. Writing $b_n = x^n/n^2$, we have

$$\left|\frac{b_{n+1}}{b_n}\right| = \frac{n^2\,|x|}{(n+1)^2}$$

which again converges to $|x|$, showing the series to be absolutely convergent if $|x| < 1$, and divergent if $|x| > 1$. On this occasion, we have absolute convergence at $x = \pm 1$, since then $\sum_1^\infty |b_n| = \sum_1^\infty 1/n^2$.

Hence, $\sum_1^\infty x^n/n^2$ converges absolutely if $|x| \leq 1$, and diverges otherwise. □

## 3.32 Exercise

Discuss the convergence and absolute convergence of the following power series for various values of $x$.

$$\text{(i)}\ \sum_1^\infty \frac{x^n}{n!} \qquad \text{(ii)}\ \sum_1^\infty \frac{(-1)^n}{n} x^{2n}$$ □



The above examples demonstrate the various possibilities that can occur within the constraints of the following general theorem on the convergence of power series.

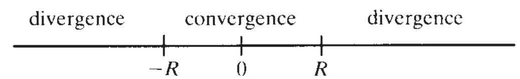
## 3.33 Theorem: Radius of convergence

For any power series $\sum_0^\infty a_n x^n$, precisely one of the following alternatives must occur.

(i) $\sum_0^\infty a_n x^n$ converges for all $x$.

(ii) $\sum_0^\infty a_n x^n$ converges only if $x = 0$.

(iii) There exists a real number $R > 0$, called the *radius of convergence*, such that $\sum_0^\infty a_n x^n$ converges if $|x| < R$ and diverges if $|x| > R$.




We describe case (i) by saying $R = \infty$, and case (ii) by saying $R = 0$. Examples of series in these categories are, respectively, $\sum_1^\infty x^n/n!$, $\sum_1^\infty n!\,x^n$. In case (iii), no general statement can be made about behaviour at $x = \pm R$. The examples 3.31 above all have $R = 1$, but exhibit different behaviour at $x = \pm 1$.

**Proof of 3.33** Is achieved by means of the following.

**Key lemma** If $\sum_0^\infty a_n x^n$ converges when $x = X$, then it converges absolutely for all $x$ satisfying $|x| < |X|$.

**Proof of key lemma** Observe that, if $|x| < |X|$, then

$$|a_n x^n| = |a_n X^n| \left|\frac{x}{X}\right|^n \leq M \left|\frac{x}{X}\right|^n,$$

where $M = \sup_{n\geq 0} |a_n X^n|$, which exists since $(a_n X^n)_{n\geq 0}$ is a null sequence (by 3.10) and is therefore bounded (by 2.29). Hence $\sum_0^\infty |a_n x^n|$ converges by comparison with $\sum_0^\infty |x/X|^n$, a geometric series with common ratio $|x/X| < 1$. □

**Proof of 3.33** Suppose we are not in cases (i) or (ii). We shall show that we must then be in case (iii).

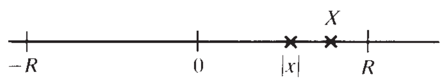
Let the set $E$ be defined as follows.

$$E = \left\{x : \sum_0^\infty a_n x^n \text{ converges}\right\}.$$

The assumption that we are not in case (i) ensures that $E$ is bounded since, if $X$ is such that $\sum_0^\infty a_nX^n$ diverges, then, by the key lemma, $\sum_0^\infty a_nx^n$ cannot converge for any $x$ satisfying $|x|>|X|$. The fact that any power series converges trivially at $x=0$ ensures that $E$ is non-empty. Therefore, by the upper bound axiom, $E$ has a supremum.

Let $R=\sup E$. The assumption that we are not in case (ii) implies that $R>0$, since, if $X\neq 0$ is such that $\sum_0^\infty a_nX^n$ converges, then, by the key lemma, $\sum_0^\infty a_nx^n$ also converges for all $|x|>|X|$, showing $R\geqslant|X|$.

Suppose $|x|<R$. Then there must exist $X$ satisfying $|x|<X\leqslant R$ such that $\sum_0^\infty a_nX^n$ converges (since $R=\sup E$). Therefore, by the key lemma, $\sum_0^\infty a_nx^n$ converges (absolutely).




Suppose $|x|>R$. Then, by the key lemma, convergence of $\sum_0^\infty a_nx^n$ for this $x$ would imply convergence of $\sum_0^\infty a_nX^n$, where


$X=\frac{1}{2}(|x|+R)>R$, which would contradict the definition of $R$. Therefore $\sum_0^\infty a_nx^n$ must diverge. □

Observe that we have proved $\sum_0^\infty a_nx^n$ converges *absolutely* for $|x|<R$. Observe also that, when we are in the case $R=\infty$, we have *absolute* convergence of $\sum_0^\infty a_nx^n$ for all $x$ (by the key lemma).

## 3.34 Exercises

Find the radius of convergence of the following power series.

$$\text{(i)}\ \sum_0^\infty (2^n+3^n)x^n \qquad \text{(ii)}\ \sum_1^\infty \frac{(2n)!}{(n!)^2}x^n$$

*Hint.* Use the ratio test. □

The reader may have noticed a general lack of theorems about arithmetical operations on series. The only results in this direction we have so far mentioned are 3.8 (i) and (ii) concerning addition and constant multiples of series. We shall now attempt to extend this somewhat limited repertoire by considering the possibility of multiplying series.

Suppose we have two series $\sum_1^\infty a_n$, $\sum_1^\infty b_n$ and we try to multiply



them together. If we assume an infinite distributive law, we ought to be able to say

$$\left(\sum_{m=1}^\infty a_m\right)\left(\sum_{n=1}^\infty b_n\right)=\sum_{m,n=1}^\infty a_mb_n.$$

However, the question immediately arises as to what order the terms $a_mb_n$ of the series $\sum_{m,n=1}^\infty a_mb_n$ should be taken in.

If we were to assume an infinite commutative law of addition, then the ordering of the terms would be immaterial. Unfortunately, no such law exists as the following example shows.

Consider the alternating harmonic series

$$\sum_1^\infty \frac{(-1)^{n-1}}{n}=1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\cdots,$$

which we know to be convergent by the alternating series test (3.26). Let its sum be $s$. Suppose we rearrange the terms taking two negative terms for each positive term. We get

$$\begin{aligned}1-\tfrac{1}{2}-\tfrac{1}{4}+\tfrac{1}{3}-\tfrac{1}{6}-\tfrac{1}{8}+\cdots&=(1-\tfrac{1}{2})-\tfrac{1}{4}+(\tfrac{1}{3}-\tfrac{1}{6})-\tfrac{1}{8}+\cdots\\&=\tfrac{1}{2}-\tfrac{1}{4}+\tfrac{1}{6}-\tfrac{1}{8}+\cdots\\&=\tfrac{1}{2}(1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\cdots)\\&=\tfrac{1}{2}s.\end{aligned}$$

Observe that reordering the terms has changed the sum! Perhaps it is not so strange after all that theorems on arithmetical manipulation of series appear to be objects of great rarity.

Whilst the situation is bleak in general, there is a class of series for which rearrangement and multiplication can be done with impunity, namely the *absolutely* convergent series. Observe that the counter-example above involves a *conditionally* convergent series. Indeed the term 'conditionally' convergent originates in the fact that the sum of such series, indeed their very convergence, is conditional on the order in which their terms are taken.

## 3.35 Theorem

If $\sum_1^\infty a_n$ is absolutely convergent, and if $\sum_1^\infty b_n$ is a rearrangement, i.e. the same terms but taken in a different order, then $\sum_1^\infty b_n$ is also absolutely convergent and has the same sum. □

**Proof** We assume first that $a_n\geqslant 0$ for all $n$. For each $N$, we can choose $N'$

such that $b_1, b_2, \ldots, b_N$ all occur among $a_1, a_2, \ldots, a_{N'}$. Therefore

$$\sum_1^N b_n \leqslant \sum_1^{N'} a_n \leqslant \sum_1^\infty a_n.$$

Hence $\sum_1^N b_n$ is bounded above, and so converges. Also

$$\sum_1^\infty b_n \leqslant \sum_1^\infty a_n. \qquad (2.33)$$

However, reversing the roles of $a_n$, $b_n$ gives similarly

$$\sum_1^\infty a_n \leqslant \sum_1^\infty b_n.$$

Hence

$$\sum_1^\infty a_n = \sum_1^\infty b_n.$$

We now drop the assumption that $a_n \geqslant 0$. As in the proof of 3.24, let

$$\begin{aligned} a_n^+ &= a_n \qquad &&\text{if } a_n \geqslant 0,\\ &= 0 &&\text{otherwise,}\\ a_n^- &= |a_n| &&\text{if } a_n \leqslant 0,\\ &= 0 &&\text{otherwise,}\end{aligned}$$

and let $b_n^+$, $b_n^-$ be defined similarly. Then $\sum_1^\infty b_n^+$, $\sum_1^\infty b_n^-$ are the corresponding rearrangements of $\sum_1^\infty a_n^+$, $\sum_1^\infty a_n^-$, which are both convergent series of positive terms (since $\sum_1^\infty a_n$ is absolutely convergent). Therefore, by what we have already proved, $\sum_1^\infty b_n^+$, $\sum_1^\infty b_n^-$ both converge and $\sum_1^\infty b_n^+ = \sum_1^\infty a_n^+$, $\sum_1^\infty b_n^- = \sum_1^\infty a_n^-$. Hence $\sum_1^\infty b_n$ is absolutely convergent, since

$$|b_n| = b_n^+ + b_n^-,$$

and its sum is given by

$$\begin{aligned}\sum_1^\infty b_n &= \sum_1^\infty b_n^+ - \sum_1^\infty b_n^-\\ &= \sum_1^\infty a_n^+ - \sum_1^\infty a_n^-\\ &= \sum_1^\infty a_n.\end{aligned}$$



## 3.36 Theorem

If $\sum_{m=1}^\infty a_m$, $\sum_{n=1}^\infty b_n$ are absolutely convergent, and if $\sum_{p=1}^\infty c_p$ is the series obtained by taking the products $a_m b_n$ is any order, then $\sum_{p=1}^\infty c_p$ is absolutely convergent and

$$\sum_{p=1}^\infty c_p = \Big(\sum_{m=1}^\infty a_m\Big)\Big(\sum_{n=1}^\infty b_n\Big).$$

**Proof** Again, we assume firstly that $a_m \geqslant 0$, $b_n \geqslant 0$. Let the products $a_m b_n$ be written in an infinite matrix as shown below.

$$\begin{matrix} a_1b_1 & a_1b_2 & a_1b_3\cdots\\ a_2b_1 & a_2b_2 & a_2b_3\cdots\\ a_3b_1 & a_3b_2 & a_3b_3\cdots\\ \vdots & \vdots & \vdots \end{matrix}$$

And let, in the first instance, $\sum_{p=1}^\infty c_p$ represent $\sum_{m,n=1}^\infty a_m b_n$ taken in the following order.

$$\begin{matrix} c_1 & c_4 & c_9\cdots\\ c_2 & c_3 & c_8\cdots\\ c_5 & c_6 & c_7\cdots\\ \vdots & \vdots & \vdots \end{matrix}$$

Then, for each $N$, we have

$$\sum_{p=1}^{N^2} c_p = \Big(\sum_{m=1}^N a_m\Big)\Big(\sum_{n=1}^N b_n\Big).$$

Taking limits, this shows that a subsequence of the partial sums of $\sum_{p=1}^\infty c_p$ converges to $(\sum_{m=1}^\infty a_m)(\sum_{n=1}^\infty b_n)$. Therefore, since $c_p \geqslant 0$, the whole sequence does, i.e.

$$\sum_{p=1}^\infty c_p = \Big(\sum_{m=1}^\infty a_m\Big)\Big(\sum_{n=1}^\infty b_n\Big).$$

By 3.35 the same result holds for any rearrangement of $\sum_{p=1}^\infty c_p$.

Suppose now that we drop the assumption that $a_m \geqslant 0$, $b_n \geqslant 0$. The above argument applied to $\sum_m |a_m|$, $\sum_n |b_n|$, $\sum_p |c_p|$ shows that $\sum_p c_p$ is absolutely convergent. Therefore $\sum_p c_p$ is convergent, and its sum is independent of the order in which its terms are taken. If we adopt the special ordering indicated at the beginning of the proof, we still have

$$\sum_{p=1}^{N^2} c_p = \left(\sum_{m=1}^{N} a_m\right)\left(\sum_{n=1}^{N} b_n\right),$$

which, on taking limits, gives

$$\sum_{p=1}^{\infty} c_p = \left(\sum_{m=1}^{\infty} a_m\right)\left(\sum_{n=1}^{\infty} b_n\right).$$ □

## 3.37 Application: Multiplying power series

Suppose we have two power series $\Sigma_0^\infty a_n x^n$, $\Sigma_0^\infty b_n x^n$. If we formally multiply them, we obtain a third power series $\Sigma_0^\infty c_n x^n$, where

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0.$$

This power series $\Sigma_0^\infty c_n x^n$ is called the *Cauchy product* of $\Sigma_0^\infty a_n x^n$, $\Sigma_0^\infty b_n x^n$. If $\Sigma_0^\infty a_n x^n$, $\Sigma_0^\infty b_n x^n$ both have radius of convergence $R$, then they converge absolutely for all $|x| < R$ (see 3.33); therefore, by 3.36, $\Sigma_0^\infty c_n x^n$ must have radius of convergence at least $R$, and

$$\sum_0^\infty c_n x^n = \left(\sum_0^\infty a_n x^n\right)\left(\sum_0^\infty b_n x^n\right)$$

for all $|x| < R$.

For example, the Cauchy product of $\Sigma_0^\infty x^n$ with itself is $\Sigma_0^\infty (n+1)x^n$, so we have

$$\sum_0^\infty (n+1)x^n = \frac{1}{(1-x)^2}$$

for all $|x| < 1$. It is interesting to observe that this result can be obtained in two other ways. One can either differentiate the formula

$$\sum_0^\infty x^n = \frac{1}{1-x},$$

or one can expand $(1-x)^{-2}$ by the binomial theorem. Neither of these alternative approaches can be justified rigorously at this stage of course. □

## 3.38 Miscellaneous exercises

1. Which of the following series converge?

(i) $\sum_1^\infty \frac{1}{n^2+1}$ (ii) $\sum_1^\infty \frac{1}{n-\frac{1}{2}}$



(iii) $\sum_1^\infty \frac{1+2^n}{1+3^n}$ (iv) $\sum_1^\infty \frac{1}{3^n-2^n}$

(v) $\sum_1^\infty \left(\frac{9}{10}+\frac{1}{n}\right)^n$ (vi) $\sum_1^\infty (\sqrt{(n+1)}-\sqrt{n})$

2. Which of the following series converge absolutely or conditionally?

(i) $\sum_1^\infty (-1)^n \frac{1}{n^2+1}$ (ii) $\sum_1^\infty (-1)^n \frac{n}{n^2+1}$

(iii) $\sum_1^\infty (-1)^n \frac{n^2}{n^2+1}$ (iv) $\sum_1^\infty (-1)^n(\sqrt{(n+1)}-\sqrt{n})$

(v) $\sum_2^\infty \frac{(-1)^n}{n+(-1)^n}$ (vi) $\sum_2^\infty \frac{(-1)^n}{n^2+(-1)^n}$

3. Find the radius of convergence of the following power series.

(i) $\sum_0^\infty \frac{1+2^n}{1+3^n} x^n$ (ii) $\sum_0^\infty 2^{\sqrt{n}} x^n$

(iii) $\sum_0^\infty (\sqrt{(n+1)}-\sqrt{n})x^n$ (iv) $\sum_1^\infty \frac{(3n)!}{n!\,(2n)!} x^n$

(v) $\sum_1^\infty \frac{n^n}{n!} x^n$ (vi) $\sum_1^\infty \left(\frac{1}{2}+\frac{1}{n}\right)^n x^n$

4. Show that, if for each $n \geq 1$, $d_n$ is an integer between 0 and 9 inclusive, then the series $\Sigma_1^\infty d_n/10^n$ converges, and its sum satisfies

$$0 \leq \sum_1^\infty \frac{d_n}{10^n} \leq 1.$$

Show conversely that, given any real number $x$ satisfying $0 \leq x \leq 1$, there exists a sequence $(d_n)_{n\geq 1}$ as above such that $\Sigma_1^\infty d_n/10^n = x$.

5. Prove that if $\alpha$ is a positive integer, and if $0 < x < 1$ is real, then the sequence $(n^\alpha x^n)_{n\geq 1}$ is null. *Hint*: Show $\Sigma_1^\infty n^\alpha x^n$ converges by the ratio test. Find the limit of the sequence

$$\left(\frac{2^n+n^3}{3^n+n^2}\right)_{n\geq 1}.$$

Find the radius of convergence of the power series

$$\sum_1^\infty \frac{x^n}{2^n - n}.$$

6. Discuss the convergence and absolute convergence of the series

$$1 - \frac{1}{2} + \frac{1}{\sqrt{3}} - \frac{1}{4} + \frac{1}{\sqrt{5}} - \frac{1}{6} + \cdots$$

N.B. Neither the ratio test nor the alternating series test is applicable here.

7. (Condensation test) Show that, if $0 \leqslant a_{n+1} \leqslant a_n$ for all $n$, then the series $\sum_1^\infty a_n$, $\sum_1^\infty 2^n a_{2^n}$ either both converge or both diverge.

(This is a refinement of the method of 3.4 and 3.6.)

Discuss the convergence of $\sum_1^\infty 1/n^\alpha$ for various values of $\alpha$.

8. ($n$th root test) Show that, if $a_n \geqslant 0$, and if $\sqrt[n]{a_n}$ converges to a limit $l$, then $\sum_1^\infty a_n$ converges if $l < 1$, and diverges if $l > 1$. *Hint*: Compare with a geometric series. (cf. Proof of the ratio test (3.21).)

Use the $n$th root test to confirm your answer to question 3, part (vi).

What can you say if $l = 1$?

9. Show that, if $a_n > 0$, $b_n > 0$ and

$$\frac{a_{n+1}}{a_n} \leqslant \frac{b_{n+1}}{b_n}$$

for all $n \geqslant 1$, and if $\sum_1^\infty b_n$ converges, then also $\sum_1^\infty a_n$ converges.

Investigate the convergence of

$$\sum_1^\infty \frac{(2n)!}{(n!)^2} x^n$$

at $x = \pm\frac{1}{4}$.

10. Given that

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots = s,$$

show that

(i) $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \frac{3}{2}s$,

(ii) $1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$ diverges to infinity.

*Hints*: (i) Observe that e.g.

$$\begin{aligned}(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{1}{8}) + (\tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{6} - \tfrac{1}{8})\\ = 1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4}.\end{aligned}$$

(ii) Use the A.M./H.M. inequality to show that the $n$th block of positive terms

$$\frac{1}{n^2 - n + 1} + \cdots + \frac{1}{n^2 + n - 1} > \frac{1}{n}.$$



11. Show that the Cauchy product of $\sum_1^\infty x^n/n$ with itself is

$$\sum_1^\infty \frac{2}{n+1}\left(1 + \tfrac{1}{2} + \cdots + \frac{1}{n}\right)x^{n+1}.$$

Show that this series is conditionally convergent at $x = -1$, and deduce that its radius of convergence is 1. *Hint*: Let

$$a_n = \frac{1}{n+1}\left(1 + \tfrac{1}{2} + \cdots + \frac{1}{n}\right).$$

Show that $a_n \geqslant a_{n+1}$ and

$$a_{2^n} < \frac{n + \frac{1}{2}}{2^n + 1}$$

by the method of 3.6. Deduce $(a_n)$ is null and use the alternating series test.

12. Show that the Cauchy product of $\sum_1^\infty x^n/\sqrt{n}$ with itself is $\sum_1^\infty c_n x^{n+1}$, where

$$c_n = \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{[2(n-1)]}} + \frac{1}{\sqrt{[3(n-2)]}} + \cdots + \frac{1}{\sqrt{n}}.$$

Show this series diverges at $x = 1$, and deduce that its radius of convergence is 1. *Hint*: Show that $c_n \geqslant 2$ for all $n$.

13. Prove that, if $\sum_1^\infty a_n$ converges, and if $\sum_1^\infty b_n$ is obtained from $\sum_1^\infty a_n$ by altering a finite number of terms, then $\sum_1^\infty b_n$ also converges, but its sum may differ from that of $\sum_1^\infty a_n$.

Hence give an alternative proof of 3.19.

14. Prove that, if $\sum_1^\infty a_n$ converges, and if $\sum_1^\infty b_n$ is obtained from $\sum_1^\infty a_n$ by grouping the terms, e.g.,

$$\begin{aligned}\sum_1^\infty b_n &= (a_1 + a_2) + (a_3 + a_4 + a_5 + a_6) + (a_7) + \cdots\\ &= b_1 + b_2 + b_3 + \cdots,\end{aligned}$$

then $\sum_1^\infty b_n$ converges, and $\sum_1^\infty b_n = \sum_1^\infty a_n$. *Hint*: Use 2.37.

15. Given that $\sum_1^\infty a_n$ converges absolutely, prove that

$$\left|\sum_1^\infty a_n\right| \leqslant \sum_1^\infty |a_n|.$$

*Hint*: Consider the proof of 3.24.