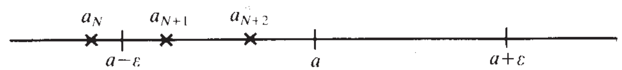
## 2.1 Definition

We say the sequence $(a_n)$ *converges* to the *limit* $a$ if, given any real number $\varepsilon > 0$, there exists a positive integer $N$ such that

$$|a_n - a| < \varepsilon$$

for all $n > N$. □




In words, $a_n$ must approximate $a$ to within $\varepsilon$ if $n$ is larger than $N$.

Observe that we must allow ourselves to be *given* $\varepsilon$ by some notional second person, who is at liberty to choose $\varepsilon$ as small as he (or she) pleases, subject only to the condition that $\varepsilon > 0$. Once this second person has committed himself to a particular value for $\varepsilon$, we must then choose $N$ which ensures this degree of accuracy for *every* $n$ beyond $N$.

It is as if we were playing a kind of game in which one player nominates $\varepsilon$, and the other nominates $N$. The rules of the game are simply that the $\varepsilon$-player goes first, and the $N$-player wins if he can find a suitable $N$, and loses if he can't. In terms of this game, the sequence $(a_n)$ converges to $a$ if the $N$-player can produce a strategy which gives him a win whatever the $\varepsilon$-player does.

For example, consider the sequence $(1/n)$. We can show $(1/n)$ converges to 0 within the terms of the above definition as follows. If $\varepsilon > 0$ is given, then the strategy of choosing $N \geqslant 1/\varepsilon$ (possible by the Archimedean axiom (1.30)) ensures a win for the $N$-player since, if $n > N$, then

$$\frac{1}{n} < \frac{1}{N} \leqslant \varepsilon.$$

## 2.2 Exercises

Give a strategy for choosing $N$ (in terms of $\varepsilon$) such that $a_n < \varepsilon$ for all $n > N$, where $a_n$ is as follows:

(i) $1/n^2$, (ii) $1/2^n$, (iii) $1/\sqrt{n}$. □

Having given a rigorous definition for the limit of a sequence (2.1), we must now construct a theory of convergence with the ultimate object of being able to tell at a glance which sequences converge to which points. We shall concentrate firstly on sequences



which converge to zero. This will make the early results easy to state and prove. Later we shall allow sequences to converge to other points. We may even allow them to diverge.

We shall investigate first the interaction between convergence and the operations of arithmetic.

## 2.3 Definition

We shall call $(a_n)$ a *null sequence* if $(a_n)$ converges to 0. □

Hence, for example, $(1/n)$ is a null sequence.

## 2.4 Theorem

If $(a_n)$, $(b_n)$ are null sequences, then the sequence $(a_n + b_n)$ formed by adding corresponding terms is also null.

**Proof** In accordance with the rules of the $\varepsilon$–$N$ game described above, we must firstly allow ourselves to be given $\varepsilon > 0$, and we must then devise a strategy for choosing $N$ which ensures that

$$|a_n + b_n| < \varepsilon$$

for all $n > N$.

Now, we know that $(a_n)$, $(b_n)$ are null, so it follows that $N'$, $N''$ can be chosen such that $|a_n| < \frac{1}{2}\varepsilon$ for all $n > N'$, and $|b_n| < \frac{1}{2}\varepsilon$ for all $n > N''$. Hence, if we take $N = \max\{N', N''\}$, then we have

$$\begin{aligned} |a_n + b_n| &\leqslant |a_n| + |b_n| \qquad (1.14) \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon \\ &= \varepsilon \end{aligned}$$

for all $n > N$. □

## 2.5 Theorem

If $(a_n)$, $(b_n)$ are null, then so is the sequence $(a_n b_n)$ formed by multiplying corresponding terms.

**Proof** The proof is similar to that of 2.4. Given $\varepsilon > 0$, we now choose $N'$, $N''$ such that $|a_n| < \sqrt{\varepsilon}$ for all $n > N'$, and $|b_n| < \sqrt{\varepsilon}$ for all $n > N''$, and take $N = \max\{N', N''\}$. We then have

$$|a_n b_n| = |a_n|\,|b_n| < \sqrt{\varepsilon}\cdot\sqrt{\varepsilon} = \varepsilon$$

for all $n > N$. □

## 2.6 Definition

The sequence $(a_n)$ is said to be *bounded* if there exists $M$ such that $|a_n| \leq M$ for all $n$. (See 1.34, question 5.) □

For example, $(1/n)$ is bounded, $(n)$ is not.

## 2.7 Theorem

If $(a_n)$ is null and $(b_n)$ is bounded, then $(a_n b_n)$ is null. □

**Proof** Let $M$ be such that $|b_n| \leq M$ for all $n$. Then, if $\varepsilon > 0$ is given, choose $N$ such that $|a_n| < \varepsilon/M$ for all $n > N$. It then follows that also

$$|a_n b_n| = |a_n|\,|b_n| < \frac{\varepsilon}{M}\cdot M = \varepsilon$$

for all $n > N$.

## 2.8 Corollary

If $(a_n)$ is null and $C$ is constant, then $(Ca_n)$ is null. □

**Proof** Put $b_n = C$ for all $n$ in 2.7. □

## 2.9 Exercises

Show the following sequences are null.

(i) $\dfrac{1}{n} + \dfrac{2}{n^2} - \dfrac{3}{n^3}$ (ii) $\dfrac{(-1)^n}{n}$ □

Having described the facts of life as they apply to null sequences (2.4 to 2.8), we now proceed to sequences with non-zero limits.



## 2.10 Theorem

$(a_n)$ converges to $a$ if and only if the sequence $(a_n - a)$ is null. □

**Proof** This is immediate from the definition of convergence (2.1). □

## 2.11 Corollaries

If $(a_n)$, $(b_n)$ converge to $a$, $b$ respectively, then

(i) $(a_n + b_n)$ converges to $a + b$,
(ii) $(a_n - b_n)$ converges to $a - b$,
(iii) $(a_n b_n)$ converges to $ab$,
(iv) $(a_n/b_n)$ converges to $a/b$, provided $b_n \neq 0$ for all $n$, and $b \neq 0$. □

**Proofs** (i) We have to show the sequence whose $n$th term is $(a_n + b_n) - (a + b)$ is null. In fact,

$$(a_n + b_n) - (a + b) = (a_n - a) + (b_n - b)$$

so is null by 2.4.

(ii) is similar (use 2.4 and 2.8).

(iii) Observe that

$$a_n b_n - ab = (a_n - a)(b_n - b) + a(b_n - b) + b(a_n - a),$$

so is null by 2.4, 2.5 and 2.8.

(iv) Observe that

$$\frac{a_n}{b_n} = a_n \frac{1}{b_n},$$

so it will be sufficient (by (iii)) to show that $(1/b_n)$ converges to $1/b$. Now

$$\frac{1}{b_n} - \frac{1}{b} = \frac{b - b_n}{b_n b},$$

so (iv) will follow by 2.7 and 2.8, if we can show that the sequence $(1/b_n)$ is bounded.

This can be achieved as follows. We know there must be an $N$ such that

$$|b_n - b| < \tfrac{1}{2}|b|$$

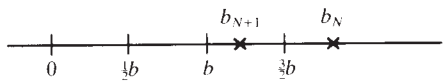
for all $n > N$ (by taking $\varepsilon = \frac{1}{2}|b|$ in the definition 2.1). It follows that

$$\tfrac{1}{2}b < b_n < \tfrac{3}{2}b$$

if $b > 0$, or

$$\tfrac{3}{2}b < b_n < \tfrac{1}{2}b$$

if $b < 0$. The picture for $b > 0$ is as shown in the diagram. Therefore, in all cases, we have $|b_n| > \frac{1}{2}|b|$ for all $n > N$.




Hence, if we take

$$M = \max\left\{\frac{2}{|b|}, \frac{1}{|b_1|}, \ldots, \frac{1}{|b_N|}\right\},$$

then we have $|1/b_n| \leq M$ for all $n$. □

## 2.12 Worked example

Find the limit of the sequence whose $n$th term is

$$\frac{n^2 + 2n + 3}{4n^2 + 5n + 6}.$$

The technique is to write

$$\frac{n^2 + 2n + 3}{4n^2 + 5n + 6} = \frac{1 + 2/n + 3/n^2}{4 + 5/n + 6/n^2},$$

which converges to

$$\frac{1 + 2.0 + 3.0^2}{4 + 5.0 + 6.0^2} = \tfrac{1}{4},$$

by 2.11. □

## 2.13 Exercises

Find limits of the following sequences.

(i) $\dfrac{n+1}{n+2}$ (ii) $\dfrac{n+1}{n^2+1}$ (iii) $\dfrac{2+n^2}{2-n^2}$. □

Our next task will be to consider the interaction between convergence and inequalities.



## 2.14 Theorem

If $a_n \leq b_n$ for all $n$, and $(a_n)$, $(b_n)$ converge to $a$, $b$ respectively, then we must have $a \leq b$. □

**Proof** We shall show that the contrary assumption $a > b$ leads to a contradiction. In fact, if $a > b$, then there would exist $N'$, $N''$ such that

$$|a_n - a| < \tfrac{1}{2}(a - b)$$

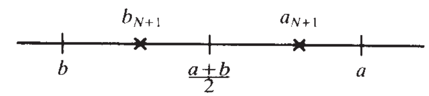
for all $n > N'$, and

$$|b_n - b| < \tfrac{1}{2}(a - b)$$

for all $n > N''$. Therefore

$$a_n > \tfrac{1}{2}(a + b) > b_n$$

for all $n > N = \max\{N', N''\}$, which provides the required contradiction. □




One might expect that if we assume the *strict* inequality $a_n < b_n$ for all $n$, then we can conclude that strict inequality $a < b$ also holds in the limit. It turns out, however, that we can only conclude that $a \leq b$. For example, consider the case $a_n = 0$, $b_n = 1/n$. We have $a_n < b_n$ for all $n$, but $a = b = 0$.

The moral of the tale is that, if we have an inequality which holds for all values of a positive integer variable $n$, then it is legitimate to take limits, *provided* we replace $<$ by $\leq$ where applicable.

## 2.15 Theorem: Uniqueness of limits

A sequence $(a_n)$ cannot converge to two different limits $a \neq b$ simultaneously.

**Proof** If $(a_n)$ converges to $a$ and $b$, then, taking $a_n = b_n$ in 2.14, we obtain $a \leq b$ and $b \leq a$. Hence $a = b$. □

## 2.16 Theorem: Sandwich principle

If $(a_n)$, $(b_n)$, $(c_n)$ are three sequences such that

$$a_n \leq b_n \leq c_n$$

for all $n$, and $(a_n)$, $(c_n)$ both converge to the same limit $l$, then also $(b_n)$ converges to $l$.

**Proof** Suppose $\varepsilon > 0$ is given. Then we can choose $N'$, $N''$ such that

$$|a_n - l| < \varepsilon$$

for all $n > N'$, and

$$|b_n - l| < \varepsilon$$

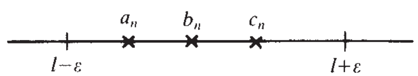
for all $n > N''$. Therefore, if $N = \max\{N', N''\}$, we have

$$l - \varepsilon < a_n \leq b_n \leq c_n < l + \varepsilon,$$

and hence

$$|b_n - l| < \varepsilon,$$

for all $n > N$. □




## 2.17 Application

$(1/2^n)$ is null. In fact, we have

$$0 < \frac{1}{2^n} < \frac{1}{n}$$

for all $n$, and $(0)$, $(1/n)$ both converge to 0. □

More generally, we can prove the following.

## 2.18 Theorem

For any fixed $x$ satisfying $|x| < 1$, the sequence $(x^n)_{n \geq 1}$ is null. □

**Proof**
**Case 1**
**$0 < x < 1$** In this case we can write

$$\frac{1}{x} = 1 + h$$

where $h > 0$. Therefore, by Bernoulli's inequality (1.12), we have

$$\frac{1}{x^n} = (1 + h)^n \geq 1 + nh,$$



which gives

$$0 < x^n \leq \frac{1}{1 + nh}.$$

Hence $(x^n)$ is null by the sandwich principle.

**Case 2**
**$-1 < x < 0$** In this case we can write $x = -u$ where $0 < u < 1$. Therefore $x^n = \pm u^n$, and so

$$-u^n \leq x^n \leq u^n.$$

Hence, again, $(x^n)$ is null by the sandwich principle.

**Case 3**
**$x = 0$** is of course trivial. □

## 2.19 Exercises

Use the sandwich principle to show that the following sequences are null.

$$\text{(i) } \frac{1}{n!} \qquad \text{(ii) } \frac{2^n + 3^n}{4^n + 5^n}$$ □

We now turn our attention to divergent sequences. There are essentially two ways in which a sequence can diverge. It either 'diverges to infinity' or else it 'oscillates'. The definitive example of divergence to infinity is the sequence $(n)$, and the definitive example of an oscillating sequence is $(-1)^n$.

To say a sequence $(a_n)$ diverges to infinity means that its $n$th term becomes indefinitely large as $n$ gets large. More precisely, $a_n$ can be made as large as anyone might care to specify, provided we take $n$ large enough. We are again in a two-handed game situation, except that now our opponent nominates a number which can be as *large* as he pleases, and then we have to find $N$ which makes $a_n$ this large for $n > N$. The formal definition is as follows.

## 2.20 Definition

The sequence $(a_n)$ *diverges to infinity* if, given any $C > 0$, there exists $N$ such that $a_n > C$ for all $n > N$. □

Restricting $C$ to be positive is a matter of convenience, and makes no essential difference since, if $N$ can be chosen corresponding to $C = 1$, say, then this same $N$ will serve equally well for any $C < 1$, including all $C \leq 0$.

## 2.21 Exercises

Find $N$ (in terms of $C$) such that $a_n > C$ for all $n > N$, where $a_n$ is as follows.

(i) $n^2$ (ii) $\sqrt{n}$ (iii) $2^n$

Hence deduce that the sequences with these $n$th terms all diverge to infinity. □

One can define a concept of divergence to *minus* infinity in a similar fashion. One simply requires terms to become indefinitely large *negative,* the further one moves along the sequence. The corresponding game involves the first player choosing $C$ as large negative as he pleases, and the second player then choosing $N$ which makes all the terms of the sequence beyond the $N$th become larger negative than $C$. The formal definition is as follows.

## 2.22 Definition

The sequence $(a_n)$ *diverges to minus infinity* if, given any $C < 0$, there exists $N$ such that $a_n < C$ for all $n > N$. □

As for divergence to (plus) infinity, the restriction on $C$ of being negative is for mere convenience, and doesn't affect the game in any essential way.

Examples of sequences which diverge to minus infinity are $(-n)$, $(-n^2)$, $(-\sqrt{n})$ etc. In fact, for any sequence $(a_n)$ which diverges to infinity, the sequence $(-a_n)$ clearly diverges to minus infinity.

Having formally defined convergence and divergence to plus and minus infinity, we shall cover all other cases with the following definition.

## 2.23 Definition

The sequence $(a_n)$ will be said to *oscillate* if it neither converges, nor diverges to either plus or minus infinity. □

## 2.24 Example

$(-1)^n$ oscillates.

**Proof** Suppose $(-1)^n$ were to converge to $a$. Then there would exist $N$ such that

$$|(-1)^n - a| < 1$$

for all $n > N$. If we take *even* $n > N$, we obtain

$$|1 - a| < 1,$$

which implies $a > 0$, but, if we take *odd* $n > N$, we obtain

$$|-1 - a| < 1,$$

which implies $a < 0$. So we have a contradiction.

Clearly $(-1)^n$ cannot diverge to plus or to minus infinity. □

Oscillating sequences are in two minds (at least) about what they wish to do. They may try to converge to more than one limit, as in the above example. They may even try to diverge to infinity as well.

There is a theory for sequences which diverge to infinity, analogous to the theory so far developed for convergent sequences. For instance, we have the following.

## 2.25 Theorem

(i) If $(a_n)$ diverges to infinity and $(b_n)$ is bounded below, then $(a_n + b_n)$ diverges to infinity.

(ii) If $(a_n)$ diverges to infinity and there exists $m$ such that $b_n \geq m > 0$ for all $n$, then $(a_n b_n)$ diverges to infinity.

(iii) *Open sandwich principle* If $(a_n)$ diverges to infinity and $a_n \leq b_n$ for all $n$, then also $(b_n)$ diverges to infinity.

**Proofs** (i) Suppose $C > 0$ is given. Choose $N$ such that

$$a_n > C - m$$

for all $n > N$, where $m = \inf(b_n)$ (see 1.29). Then

$$a_n + b_n > (C - m) + m = C$$

for all $n > N$.

(ii) If $C > 0$ is given, choose $N$ such that $a_n > C/m$ for all $n > N$. Then

$$a_n b_n > \frac{C}{m} \cdot m = C$$

for all $n > N$.

(iii) Given $C > 0$, choose $N$ such that $a_n > C$ for all $n > N$. Then also $b_n \geq a_n > C$ for all $n > N$. □

## 2.26 Application

$(x^n)_{n \geqslant 1}$ diverges to infinity if $x > 1$.

**Proof** We can write $x = 1 + h$ where $h > 0$. Therefore, by Bernoulli's inequality (1.12), we have

$$x^n = (1+h)^n \geqslant 1 + nh,$$

and $(1+nh)_{n \geqslant 1}$ diverges to infinity, by 2.25, (i) and (ii). Hence $(x^n)$ diverges to infinity by 2.25 (iii). □

Observe that for $x < -1$, the sequence $(x^n)$ oscillates. In fact, it tries to diverge to plus and minus infinity simultaneously, a hopelessly ambitious project.

## 2.27 Worked example

Show that the sequence whose $n$th term is

$$\frac{n^2+4}{n+4}$$

diverges to infinity.

In fact,

$$\frac{n^2+4}{n+4} \geqslant \frac{n^2}{5n} = \frac{n}{5},$$

so the result follows by 2.25, (ii) and (iii).

## 2.28 Exercises

Show the following sequences diverge to infinity:

$$\text{(i)}\ \frac{n^3+2}{n^2+2} \qquad \text{(ii)}\ \frac{5^n+4^n}{3^n+2^n}.$$ □

It remains for us to discuss the implications of the upper bound axiom (1.26) in the theory of sequential convergence.

We have already considered bounded sequences (see 2.6), though we have yet to prove one of the most important theorems concerning bounded sequences, an omission we now hasten to repair.



## 2.29 Theorem

If $(a_n)$ converges, then $(a_n)$ is bounded. □

**Proof** Suppose $(a_n)$ converges to $a$. Then there must exist $N$ such that

$$a - 1 < a_n < a + 1$$

for all $n > N$. Therefore, if we take

$$M = \max\{a+1, a_1, a_2, \ldots, a_N\},$$
$$m = \min\{a-1, a_1, a_2, \ldots, a_N\},$$

then we have $m \leqslant a_n \leqslant M$ for all $n$. □

2.29 has no converse. A bounded sequence $(a_n)$ need not converge. For example, consider the case $a_n = (-1)^n$.

Nevertheless, 2.29 does have certain *restricted* converses, in the sense that one either assumes more or proves less. We shall show that convergence of $(a_n)$ can be proved if we assume a little more about $(a_n)$ than its mere boundedness (see 2.33). On the other hand, we shall also show that, for a general bounded sequence $(a_n)$, even though one cannot prove convergence, one can prove a slightly weaker result, which turns out to be surprisingly useful in applications (see 2.38).

## 2.30 Definition

A sequence $(a_n)$ is said to *increase* if $a_n \leqslant a_{n+1}$ for all $n$, or to *decrease* if $a_n \geqslant a_{n+1}$ for all $n$. $(a_n)$ is said to be *monotonic* if $(a_n)$ either increases (for all $n$) or decreases (for all $n$). □

## 2.31 Examples

$a_n = n$ increases, $a_n = 1/n$ decreases but $a_n = (-1)^n$ does neither, i.e., $(-1)^n$ is *not* monotonic. □

## 2.32 Exercises

Which sequences are monotonic?

(i) $n + (-1)^n$  (ii) $n^2 + (-1)^n$ □

## 2.33 Theorem

If $(a_n)$ is bounded and monotonic, then $(a_n)$ converges. □

This is the first restricted converse of 2.29,

More explicitly, we can isolate the following four possibilities.

(i) If $(a_n)$ increases and is bounded above, then $(a_n)$ converges to its supremum.

(ii) If $(a_n)$ increases but is not bounded above, then $(a_n)$ must diverge to infinity.

(iii) If $(a_n)$ decreases and is bounded below, then $(a_n)$ converges to its infimum.

(iv) If $(a_n)$ decreases and is unbounded below, then $(a_n)$ diverges to minus infinity.

Observe that monotonic sequences never oscillate. They are always single-minded in their intentions.

**Proof of 2.33** (i) Suppose that $(a_n)$ increases and is bounded above. Let $M = \sup(a_n)$. We have to show $(a_n)$ converges to $M$. If $\varepsilon > 0$ is given, then we must have

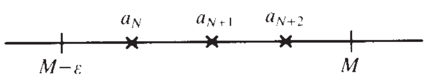
$$a_N > M - \varepsilon$$

for some $N$ (see 1.34, question 6). Therefore, for all $n > N$, we have

$$M - \varepsilon < a_N \leqslant a_n \leqslant M,$$

and hence

$$|a_n - M| < \varepsilon.$$




(ii) Suppose now that $(a_n)$ increases and is unbounded above. We now have to show $(a_n)$ diverges to infinity. If $C > 0$ is given, then we must have $a_N > C$ for some $N$ (otherwise $C$ would be an upper bound) and therefore

$$a_n \geqslant a_N > C$$

for all $n > N$.

(iii) and (iv) are proved similarly. □

The beauty of 2.33 is that it enables us to prove a sequence converges without necessarily knowing beforehand what the limit is to be.



## 2.34 Example

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

It is not at all obvious what the limit of this sequence might be. We can nevertheless prove it converges by showing it increases and is bounded above.

In fact, expanding $a_n$ by the binomial theorem, we have

$$\begin{aligned} a_n &= 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \cdots \\ &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots, \end{aligned}$$

from which it is clear that $a_n$ increases as $n$ increases. Also that

$$\begin{aligned} a_n &< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \\ &< 1 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} \\ &< 3 \end{aligned}$$

for all $n$, showing $a_n$ is bounded above. □

In fact, the limit of this sequence is well known and is e, the base of natural logarithms. Indeed, this is one of the ways of *defining* e, though it is not terribly useful in applications, or for numerical calculation of e. Alternative methods of defining e will be considered in Chapter 4.

The second restricted converse of 2.29 is concerned with subsequences, which we now define.

## 2.35 Definition

If $(a_n)$ is a sequence, then a *subsequence* of $(a_n)$ is any sequence of the form $(a_{n_r})_{r\geqslant 1}$, where $(n_r)_{r\geqslant 1}$ is a *strictly* increasing sequence of positive integers, i.e. $n_r < n_{r+1}$ for all $r$. □

## 2.36 Examples

(i) $n_r = 2r$. This gives the subsequence $a_2, a_4, a_6, a_8, \ldots$, which can be alternatively denoted by $(a_{2n})_{n\geqslant 1}$, and is called the *even* subsequence of $(a_n)$.

(ii) $n_r = 2r - 1$. This gives the *odd* subsequence $a_1, a_3, a_5, a_7, \ldots$, alternatively denoted by $(a_{2n-1})_{n\geq 1}$.

Other subsequences we shall have occasion to consider in Chapter 3 are the following.

(iii) $(a_{2^n})_{n\geq 1} = a_2, a_4, a_8, a_{16}, \ldots$.

(iv) $(a_{n^2})_{n\geq 1} = a_1, a_4, a_9, a_{16}, \ldots$. □

## 2.37 Theorem

If $(a_n)_{n\geq 1}$ converges to a limit $a$, or diverges to plus or to minus infinity, then any subsequence $(a_{n_r})_{r\geq 1}$ does the same. □

**Proof** Suppose $(a_n)_{n\geq 1}$ converges to $a$. Then, for any given $\varepsilon > 0$, we can choose $N$ such that

$$|a_n - a| < \varepsilon$$

for all $n > N$. Now choose $R$ such that $n_r > N$ for all $r > R$. This is clearly possible since $n_r$ increases strictly with $r$. It then follows that

$$|a_{n_r} - a| < \varepsilon$$

for all $r > R$, and hence $(a_{n_r})_{r\geq 1}$ converges to $a$.

If $(a_n)_{n\geq 1}$ diverges to infinity, then, for any given $C > 0$, we can choose $N$ such that $a_n > C$ for all $n > N$, and then $R$ such that $n_r > N$ for all $r > R$. Hence $a_{n_r} > C$ for all $r > R$, which proves $(a_{n_r})_{r\geq 1}$ diverges to infinity.

Divergence to minus infinity is treated similarly. □

Observe that single-minded sequences carry all their subsequences with them. Oscillatory sequences have no such power. Consider, for example, the sequence $(-1)^n$. Its even subsequence converges to 1, whilst its odd subsequence converges to $-1$.

This, incidentally, gives an alternative proof of the divergence of $(-1)^n$ (in the sense of non-convergence), since, if $(-1)^n$ were to converge to a limit $l$, then, by 2.37, all its subsequences would converge to $l$, and therefore we would have $l = 1 = -1$, which contradicts the uniqueness of limits (2.15).

We can now present our second restricted converse to 2.29.

## 2.38 Bolzano–Weierstrass theorem

Any bounded real sequence $(a_n)_{n\geq 1}$ must have a convergent subsequence $(a_{n_r})_{r\geq 1}$. □



Observe that the bounded sequence $(-1)^n$, even though not convergent, does possess convergent subsequences, e.g. the even and odd subsequences.

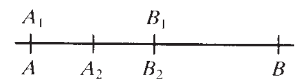
**Proof of 2.38** Since $(a_n)$ is bounded, there exist $A < B$ such that $A \leq a_n \leq B$ for all $n$. If we bisect the closed interval $[A, B]$, at least one half must contain $a_n$ for infinitely many $n$. Let $[A_1, B_1]$ be such a half. If we now bisect $[A_1, B_1]$, then, again, at least one half must contain $a_n$ for infinitely many $n$. Call such a half $[A_2, B_2]$. We can repeat this process indefinitely to obtain a nested sequence $[A_r, B_r]_{r\geq 1}$ of closed subintervals of $[A, B]$ with the following properties.

(i) $A \leq A_1 \leq A_2 \leq \cdots \leq A_r < B_r \leq \cdots \leq B_2 \leq B_1 \leq B$.

(ii) $B_r - A_r = \dfrac{B - A}{2^r}$.

(iii) For every $r$, $[A_r, B_r]$ contains $a_n$ for infinitely many $n$.

For example, the sequence might begin as shown below.




Observe that $(A_r)_{r\geq 1}$ is increasing, bounded above, therefore converges to a limit $l$, and that $(B_r)_{r\geq 1}$ is decreasing, bounded below, therefore converges to a limit $l'$. (See 2.33, (i) and (iii).) Observe further, that taking limits in (ii) gives $l' - l = 0$, i.e. $l = l'$.

So, to construct a convergent subsequence $(a_{n_r})_{r\geq 1}$ of $(a_n)$, we simply choose $a_{n_r} \in [A_r, B_r]$, for each $r$, in such a way that $n_r$ strictly increases with $r$. Condition (iii) clearly ensures that this is possible. We then have

$$A_r \leq a_{n_r} \leq B_r$$

for all $r$, and therefore, by the sandwich principle (2.16), we must have $(a_{n_r})_{r\geq 1}$ convergent to $l$. □

The Bolzano–Weierstrass theorem will be seen in later chapters to have important consequences in the development of analysis, in particular the minimax theorem for continuous functions (see 4.58), which is essential for Rolle's theorem (see 5.15) and the mean value theorem (see 5.16) in the differential calculus.

## 2.39 Miscellaneous exercises

1. Discuss the convergence or divergence of the following sequences.

(i) $\left(1+\frac{1}{n}\right)^5$ (ii) $(1+\frac{1}{5})^n$ (iii) $\left(1+\frac{1}{n}\right)^{n^2}$

(iv) $\left(1+\frac{1}{n^2}\right)^n$ (v) $\left(\frac{1}{2}+\frac{1}{n}\right)^n$ (vi) $\frac{2^n+3^n}{3^n+4^n}$

(vii) $n+(-1)^n$ (viii) $n+(-1)^n n$ (ix) $n+(-1)^n n^2$

2. Prove that if $(a_n)$ converges to $a$, then $(|a_n|)$ converges to $|a|$. (Use 1.15.)
3. Prove the inequality

$$|\sqrt{a}-\sqrt{b}| \leqslant \sqrt{|a-b|}$$

for all $a \geqslant 0$, $b \geqslant 0$.

Deduce that, if $(a_n)$ converges to $a$, where $a_n \geqslant 0$ for all $n$, then $(\sqrt{a_n})$ converges to $\sqrt{a}$.

Find the limit of the sequence whose $n$th term is $\sqrt{(n+1)}-\sqrt{n}$.
4. Prove that, if $(a_n)$ converges to $a$, then any sequence obtained from $(a_n)$ by altering a finite number of terms also converges to $a$.

Deduce the following generalized version of the sandwich principle. If $a_n \leqslant b_n \leqslant c_n$ for all $n \geqslant$ some $N$, and if $(a_n)$, $(c_n)$ both converge to $l$, then also $(b_n)$ converges to $l$.

Show the sequence $(n/2^n)$ is null. (Use 1.13.)
5. Show that, if $a_n \neq 0$ for all $n$, and if $(a_n)$ diverges to infinity, then $(1/a_n)$ is null.

Is there a converse?
6. Suppose $(a_n)$ is an increasing sequence.

Show that, if $(a_n)$ has a subsequence $(a_{n_r})_{r\geqslant 1}$ which converges to a limit $a$, then $(a_n)$ must itself converge to $a$. (Use 2.33 and 2.37.)

Show, on the other hand, that, if $(a_n)$ has a subsequence which diverges to infinity, then $(a_n)$ must diverge to infinity.
7. Given that $(1+1/n)^n$ converges to e, show that

(i) $\left(1-\frac{1}{n}\right)^n$ converges to $1/\mathrm{e}$,

(ii) $\left(1+\frac{2}{n}\right)^n$ converges to $\mathrm{e}^2$.

*Hints*: (i) Observe that

$$1-\frac{1}{n}=\frac{1}{1+\frac{1}{n-1}}.$$

(ii) Show $(1+2/n)^n$ is increasing, and that its even subsequence converges to $\mathrm{e}^2$. Now use question 6.



8. Show that, if the set $E$ is bounded above, then one can find a sequence of points $a_n \in E$ which converges to $M = \sup E$. (Use 1.34, question 6.)

Show also that there is a sequence of points $b_n \notin E$ which converges to $M$.
9. Show that, if $E$ is unbounded above, then there is a sequence of points $a_n \in E$ which diverges to infinity.
10. Show that, if the sequence $(a_n)$ is bounded and divergent, then it has two subsequences which converge to distinct limits. *Hint*: Use the Bolzano–Weierstrass theorem (2.38) twice.
11. Let $A > 0$ be fixed, and let the sequence $(a_n)$ be defined inductively as follows.

$$a_1 = 1,$$

$$a_{n+1} = \tfrac{1}{2}\left(a_n + \frac{A}{a_n}\right)$$

if $n \geqslant 1$. Show that

(i) $a_n^2 \geqslant A$ for all $n \geqslant 2$,

(ii) $a_n$ decreases for all $n \geqslant 2$.

Deduce that $(a_n)$ converges to a limit $a$.

Find the value of $a$ by taking limits in the defining equation for $(a_n)$.
12. Show that, for any $0 < a < b$,

$$a < \frac{2ab}{a+b} < \frac{a+b}{2} < b.$$

Given $0 < a_1 < b_1$, let sequences $(a_n)$, $(b_n)$ be defined by saying

$$a_{n+1} = \frac{2a_n b_n}{a_n + b_n}, \qquad b_{n+1} = \frac{a_n + b_n}{2}$$

for all $n \geqslant 1$. Prove $(a_n)$, $(b_n)$ both converge to a common limit $l$.

Find the value of $l$ when $a_1 = 1$, $b_1 = 2$.