1
The real numbers

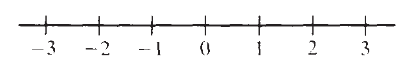
Our intention is to apply the rigorous spirit of Euclidean geometry to the subject material of the infinitesimal calculus. We shall postulate a small number of self-evident axioms and then deduce the whole subject logically from these axioms.

Since we shall be dealing with real numbers throughout, we shall present the axioms as a list of properties which we shall assume the real numbers to have. We do not wish to be too pedantic about this. Many properties are so utterly self-evident as to hardly need mentioning. We shall pay particular attention to those properties which may not be quite so familiar to a student embarking on a course in mathematical analysis for the first time.

To this end, we shall take all the *arithmetical* properties of real numbers, such as are concerned with addition, subtraction, multiplication, and division, totally for granted. We shall assume also that the integers and the rational numbers (fractions) are known and that it is unnecessary to define them.

We shall be more careful when it comes to *inequalities*. We shall spell out the axioms for inequalities in detail, and encourage the student at an early stage to obtain as much facility with inequalities as he or she can. This is because we believe that the manipulation of inequalities is at the root of analysis, and that success in analysis is not easy until confidence with inequalities is achieved.

We shall think of the real numbers as the values any continuously varying quantity may take, e.g. mass, length, time, temperature. The values may be positive or negative, and arbitrarily large either way. We shall keep in mind a geometrical picture of the real numbers as laid out along a line called the *real line* which we can imagine as calibrated with the integers as shown below.



It might be thought that one can adequately describe all the numbers which appear between integer points as rational numbers with suitably large denominators. It turns out that this is not so. In fact, the real number $\sqrt{2}$ cannot be represented as a rational number. This can be easily demonstrated as follows.

Suppose there were integers m, n such that $m^2/n^2 = 2$. We can clearly assume that m, n have no common factor. Multiplying up, we have $m^2 = 2n^2$, from which it follows that m must be even, i.e. $m = 2p$ for some integer p. Substituting for m, we obtain $4p^2 = 2n^2$ which, on cancellation, gives $2p^2 = n^2$. However, this implies that n is even, and therefore m, n have the common factor 2, which is a contradiction. Hence $\sqrt{2}$ must be *irrational.*

1.1 Exercise

Show $\sqrt{3}$, $\sqrt[3]{4}$ are irrational by a similar method. □

As we have already said, we shall assume all the arithmetical properties of real numbers without further ado. We shall also assume the principle of mathematical induction. We give a brief description in case the reader is unfamiliar with it.

Suppose we wish to prove a proposition $P(n)$, with a variable n, is true for all $n = 1, 2, 3, \ldots$ running through the positive integers. Then it is sufficient to prove that $P(1)$ is true, and that, for each $n = 1, 2, 3, \ldots, P(n)$ true implies $P(n+1)$ true.

For example, let $P(n)$ be the proposition that the sum of the first n positive integers is $\frac{1}{2}n(n+1)$. $P(1)$ is clearly true, and, if we *assume* $P(n)$ is true for any particular n, then we can deduce $P(n+1)$ is true for this n, since

$$\begin{aligned} 1+2+3+\cdots+n+(n+1) &= \tfrac{1}{2}n(n+1)+(n+1) \\ &= (\tfrac{1}{2}n+1)(n+1) \\ &= \tfrac{1}{2}(n+1)(n+2). \end{aligned}$$

Mathematical induction therefore enables us to conclude that $P(n)$ is true for all positive integers n.

Having disposed of the arithmetical aspects of the real numbers in cavalier fashion, we shall now by contrast concentrate on inequalities in depth.

The real numbers have a natural *order* along the line. Relative position in the order is expressed by saying one real number is *less than* or *greater than* another. We shall use the notation $<$ for less than, and $>$ for greater than, e.g. $2 < 3$, $5 > 1$. Expressions involving $<$ or $>$ are called *inequalities.* Inequalities have their own arithmetic which is subject to certain laws in the same way as ordinary arithmetic is. For example, we have the following.

1.2 Law of addition

If $a < b$, then, for any c, we have

$$a + c < b + c.$$ □

An inescapable consequence of the law of addition is that inequalities for negative numbers may appear at first sight to be the wrong way round. For example, if we start with the inequality $1 < 2$, and subtract 3 from both sides (case $c = -3$), we get $-2 < -1$. This is, however, consistent with the approach of saying $x < y$ if x lies to the left of y on the real line.

1.3 Law of multiplication

If $a < b$, then, for any $c > 0$, we have

$$ac < bc,$$

but, if $c < 0$, we have

$$ac > bc.$$ □

In words, multiplication of both sides of an inequality by a positive number $c > 0$ preserves the inequality, whilst multiplication by a negative number $c < 0$ *reverses* the inequality. For example, we have $1 < 2$; therefore, multiplying by 3, we get $3 < 6$, or, dividing by 3 (case $c = \frac{1}{3}$), we get $\frac{1}{3} < \frac{2}{3}$, but, if we wish to multiply by -3, we must reverse the inequality and write $-3 > -6$.

There are two other laws which may appear obvious but are none the less important for that.

1.4 Trichotomy law

For any two real numbers a, b one and only one of the three possibilities $a < b$, $a = b$, $a > b$ must occur. □

1.5 Transitive law

If $a < b$ and $b < c$, then $a < c$. □

It follows from 1.4 that, for example, if $a \not> b$ (a is not greater than

b), then either $a < b$ or $a = b$, i.e. a is less than or equal to b, and we write $a \leqslant b$. Similarly, if $a \not< b$, then we must have $a \geqslant b$.

The transitive law 1.5 can be extended to any finite number of terms. For example, if $a < b$, $b < c$, $c < d$, $d < e$, then $a < e$. A standard technique for proving an inequality $a < e$ is to find intervening points b, c, d for which the above chain of inequalities is true.

We shall now illustrate the use of the laws of inequalities just given by *solving* inequalities, which is analogous to solving equations, and *proving* inequalities, which is analogous to proving identities. The rules of course are rather different and do not always lead to the result one might expect. It is essential, however, to abide by the rules at all times. Every step in an argument must be justifiable by reference to one of the four laws given above.

1.6 Worked example

Solve the inequality

$$x + 1 < 2x + 3.$$

Subtracting 1 from both sides gives

$$x < 2x + 2.$$

Subtracting $2x$ from both sides gives

$$-x < 2.$$

Multiplying both sides by -1 gives

$$x > -2.$$

This is the answer. □

1.7 Exercise

Solve

$$\frac{3x-5}{7} > \frac{2x+5}{6}.$$ □

1.8 Another worked example

Solve

$$x^2 - 8x + 12 < 0.$$

Method 1 Factorize. We get

$$(x-6)(x-2) < 0.$$

Therefore $x-6$, $x-2$ must be of opposite sign. Either $x-6<0$ and $x-2>0$, which gives $x<6$ and $x>2$, i.e. $2<x<6$, or $x-6>0$ and $x-2<0$, which gives $x>6$ and $x<2$, which is impossible. So the answer is $2<x<6$.

Method 2 Complete the square. We get

$$(x-4)^2 - 4 < 0.$$

Adding 4 to both sides gives

$$(x-4)^2 < 4.$$

This inequality can only be satisfied if

$$-2 < x-4 < 2,$$

which, on adding 4 throughout, gives

$$2 < x < 6.$$ □

It might be asked how the second argument is justified from the laws of inequality. The argument certainly appeals to common sense, and many might feel this to be sufficient. One has to admit, however, that the reliability of one's common sense depends very much upon the extent of one's experience.

The assumption we have made in this instance is that $x^2 < a^2$ is equivalent to $-a < x < a$. This can easily be verified from the graph of $y = x^2$ (Fig. 1.1).

A rigorous justification from the laws of inequality might go like this. If $0 < x < a$, then multiplying by x gives $x^2 < ax$, and multiplying by a gives $ax < a^2$, and therefore, by the transitive law, $x^2 < a^2$.

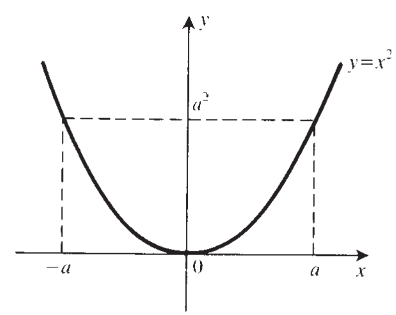


Fig. 1.1

The same argument shows that, if $x > a > 0$, then $x^2 > a^2$. A similar argument shows that $-a < x < 0$ implies $x^2 < a^2$, and that $x < -a < 0$ implies $x^2 > a^2$. This has shown that, if $-a < x < a$, then $x^2 < a^2$, whilst, if $x < -a$ or $x > a$, then $x^2 > a^2$. The equivalence of $x^2 < a^2$ and $-a < x < a$ now follows from the trichotomy law.

1.9 Exercise

Solve

$$x^2 + x - 6 > 0. \qquad \square$$

Proving inequalities can be a good deal less straightforward. Many inequalities depend on the fact that $a^2 \geqslant 0$ for all real a. For example, we have the following.

1.10 Worked example

Prove the inequality

$$\left(\frac{a+b}{2}\right)^2 \geqslant ab$$

is true for all real a, b.

Multiplying both sides by 4 gives

$$(a+b)^2 \geqslant 4ab.$$

Subtracting $4ab$ from both sides gives

$$(a+b)^2 - 4ab \geqslant 0,$$

i.e.

$$(a-b)^2 \geqslant 0,$$

which is true, therefore the original inequality is true, since the argument is reversible. $\square$

1.11 Exercise

Prove

$$(ad - bc)^2 \leqslant (a^2 + b^2)(c^2 + d^2)$$

for all real a, b, c, d. $\square$

The transitive law may also come into play as in the following.

1.12 Worked example: Bernoulli's inequality

Prove

$$(1+x)^n \geqslant 1 + nx$$

for all real $x > -1$ and all positive integers n.

We argue by induction on n. The inequality clearly holds for $n = 1$, in fact is equality for all x. Suppose the inequality is true for a particular n. We shall show this implies it is also true for $n + 1$. In fact,

$$\begin{aligned}(1+x)^{n+1} &= (1+x)^n(1+x)\\ &\geqslant (1+nx)(1+x),\end{aligned}$$

since $1 + x > 0$ on account of the fact that $x > -1$,

$$\begin{aligned}&= 1 + (n+1)x + nx^2\\ &\geqslant 1 + (n+1)x,\end{aligned}$$

since $nx^2 \geqslant 0$. Therefore, by the transitive law, we have

$$(1+x)^{n+1} \geqslant 1 + (n+1)x$$

as required. $\square$

Bernoulli's inequality is of course much easier to prove if we assume $x \geqslant 0$. In fact, we have immediately, from the binomial theorem,

$$\begin{aligned}(1+x)^n &= 1 + nx + \tfrac{1}{2}n(n-1)x^2 + \cdots + x^n\\ &\geqslant 1 + nx\end{aligned}$$

since all the other terms are $\geqslant 0$. Unfortunately this proof fails for $-1 < x < 0$.

1.13 Exercise

Prove $2^n \geqslant n^2$ for all $n \geqslant 4$. $\square$

Hint Using induction and the transitive law, the problem boils down to showing $2n^2 \geqslant (n+1)^2$ for all $n \geqslant 4$. This can either be proved directly (as in 1.8) or by taking $x = -1/n$, $n = 2$ in Bernoulli's inequality. $\square$

An important piece of notation which will be used extensively throughout this book is the so-called *modulus* or *absolute value* of x,

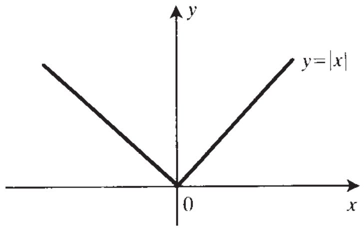


Fig. 1.2

denoted by $|x|$, and defined as follows.

$$|x| = x(x \geqslant 0),$$
$$= -x(x < 0).$$

For example $|-2| = 2$, $|0| = 0$ etc. The graph of $y = |x|$ is as shown in Fig. 1.2.

Observe that $|x| \geqslant 0$ for every x. Also that $|x| = \sqrt{x^2}$ (positive square root) and hence e.g. $|xy| = |x|\,|y|$ for all x, y.

The interaction between modulus and addition is more subtle, and is embodied in an inequality which is important enough to be given the status of a theorem.

1.14 Theorem

For all real x, y we have

$$|x + y| \leqslant |x| + |y|. \qquad \square$$

Proof Squaring both sides gives

$$|x + y|^2 \leqslant (|x| + |y|)^2,$$

which, on expanding and observing that $|x|^2 = x^2$, becomes

$$x^2 + 2xy + y^2 \leqslant x^2 + 2\,|x|\,|y| + y^2,$$

which, on cancelling and using $|xy| = |x|\,|y|$, reduces to

$$xy \leqslant |xy|,$$

which is clearly true. Hence the required inequality follows, since each of the above steps is reversible. □

1.15 Corollary

For all real x, y we have

$$\big|\,|x| - |y|\,\big| \leqslant |x - y|. \qquad \square$$

Proof Similar to 1.14. □

1.16 Exercises

Prove the following inequalities.
(i) $|ab| \leqslant \frac{1}{2}(a^2 + b^2)$.
(ii) $|a + b + c| \leqslant |a| + |b| + |c|$. □

Solving inequalities involving modulus can often be achieved by observing that $|x - y|$ represents the distance between x and y. It follows that, e.g., $|x| < A$ is equivalent to $-A < x < A$, a fact which can itself be used to solve inequalities of certain types.

1.17 Worked example

Solve $|x - 4| < 7$.
Removing the modulus sign yields

$$-7 < x - 4 < 7,$$

which, on adding 4 throughout, becomes

$$-3 < x < 11,$$

which is the answer. □

1.18 Exercises

Solve the following inequalities.
(i) $|x + 1| < 1$.
(ii) $|x + 2| < |x - 2|$. □

All the axioms or laws so far mentioned are satisfied by the rational numbers. And yet the rational numbers do not include $\sqrt{2}$. In order to ensure $\sqrt{2}$ exists as a real number we shall introduce one more axiom called the *upper bound axiom*. Before we can state this axiom, it will be necessary to set up some notation and make some definitions.

1.19 Notation

We shall write $\{x : P(x)\}$, where $P(x)$ is a proposition involving x, to mean the set of all x for which $P(x)$ is true. For example $\{x : x > 0\}$ denotes all positive numbers, $\{1/n : n = 1, 2, 3, \ldots\}$ denotes the set consisting of all reciprocals of positive integers. If $a < b$ are real numbers, we shall write

$$[a, b] = \{x : a \leqslant x \leqslant b\},$$

and call it the *closed interval* from a to b, and

$$(a, b) = \{x : a < x < b\},$$

and call it the *open interval* from a to b. If E is any set and x is any number, we shall write $x \in E$ to mean that x belongs to E. For example, $\frac{1}{2} \in [0, 1]$. □

1.20 Definition

If E is any set of real numbers, and M is another real number, we say M is an *upper bound* of E if $x \leqslant M$ for all $x \in E$. □

1.21 Examples

1 is an upper bound for the closed interval $[0, 1]$. If E is the set of all ages of living American presidents, then 120 is an upper bound for E. □

We define a *lower bound* of E similarly as any m such that $x \geqslant m$ for all $x \in E$. For example, 0 is a lower bound for both sets mentioned in 1.21.

We say E is *bounded above* if E has an upper bound, and *bounded below* if E has a lower bound. We say simply E is *bounded* if E is bounded above and below. For example, both sets of 1.21 are bounded. The set $\{x : x > 0\}$ is bounded below, but not bounded above.

1.22 Definition

We say M is the *maximum* of E, and we write $M = \max E$, if $M \in E$ and $M > x$ for all other $x \in E$, i.e. M is an upper bound of E which belongs to E.

We define the *minimum* of E, denoted by $\min E$, similarly. □

For example, $\max [0, 1] = 1$, $\min [0, 1] = 0$. However, if $E = \{1/n : n = 1, 2, 3, \ldots\}$, then clearly $\max E = 1$, but E has no minimum. This is because no point of E can be a lower bound for E since, for any particular n, we have

$$\frac{1}{n+1} < \frac{1}{n}.$$

1.23 Theorem

Every finite set has a maximum and a minimum. □

Proof This is by induction on the size of the set. If E is the singleton $\{x\}$ consisting of the single point x, then clearly $\max E = \min E = x$. Suppose the theorem is true for all sets with n points, and that E has $n + 1$ points. Let

$$E = \{x_1, x_2, \ldots, x_{n+1}\},$$

i.e. E consists of the points $x_1, x_2, \ldots, x_{n+1}$. Then

$$F = \{x_1, x_2, \ldots, x_n\}$$

has n points so has a maximum and a minimum by assumption. Let these be M, m. By the trichotomy axiom, we have $x_{n+1} > M$ or $\leqslant M$, giving respectively $\max E = x_{n+1}$ or M. The argument for $\min E$ is similar. □

1.24 Definition

If E is bounded above, then M is the *least upper bound* or *supremum* of E, denoted by $\sup E$, if M is an upper bound of E, and M is less than every other upper bound of E. □

1.25 Examples

Clearly $\sup E = \max E$ if $\max E$ exists. If E is the open interval $(0, 1)$, then $1 = \sup E$, since clearly 1 is an upper bound, and any other upper bound must be greater than 1. Observe, however, that $(0, 1)$ has no maximum.

1.26 Upper bound axiom

Every non-empty set E of real numbers which is bounded above has a supremum. □

The condition of being non-empty may seem pedantic, but in applications it is essential to know that a bounded-above set E has definitely got something in it before one can deduce the existence of a supremum from the upper bound axiom. Failure to do so can lead to fallacious arguments.

The upper bound axiom is of course very plausible, but it is in fact not at all obvious, indeed it is *false* for the rational number system, as the next theorem shows.

1.27 Theorem

$\sqrt{2}$ exists as a real number. □

Proof Let $E = \{x > 0 : x^2 < 2\}$. Clearly E is non-empty, since e.g. $1 \in E$, and E is bounded above, since e.g. 2 is an upper bound for E. Therefore, by the upper bound axiom, E has a supremum. Let $M = \sup E$. We shall show $M^2 = 2$.

Suppose $M^2 < 2$. Consider $M + 1/n$, where n is a positive integer. Observe that

$$\left(M + \frac{1}{n}\right)^2 = M^2 + \frac{2M}{n} + \frac{1}{n^2}$$

$$\leqslant M^2 + \frac{2M}{n} + \frac{1}{n},$$

for all $n \geqslant 1$,

$$= M^2 + \frac{2M + 1}{n}$$

$$< 2,$$

if

$$n > \frac{2M + 1}{2 - M^2}.$$

Therefore $M + 1/n \in E$ for such n, and of course $M + 1/n > M$, so we have a contradiction of the fact that M is an upper bound of E.

Suppose $M^2 > 2$. Then

$$\left(M - \frac{1}{n}\right)^2 = M^2 - \frac{2M}{n} + \frac{1}{n^2}$$

$$> M^2 - \frac{2M}{n}$$

$$\geqslant 2,$$

if

$$n \geqslant \frac{2M}{M^2 - 2}.$$

Therefore $M - 1/n$ is an upper bound of E for such n, and yet $M - 1/n < M$, which contradicts the fact that M is the *least* upper bound of E.

It follows from the trichotomy axiom that we must have $M^2 = 2$ as required. □

1.28 Corollary

The upper bound axiom is false in the rational number system.

This is because, if it were true, then the argument of 1.27 would show that $\sqrt{2}$ exists in the rational number system, which we know to be not so. □

1.29 Definition

If E is bounded below, we define the *greatest lower bound* or *infimum* of E, denoted by inf E, to be m such that m is a lower bound of E, and m is greater than any other lower bound of E. □

The lower bound axiom then states that any non-empty set E of real numbers which is bounded below has an infimum. We shall take the lower bound axiom as axiomatic, though it can be proved as a consequence of the upper bound axiom (see Exercise 7 at the end of the chapter).

We finish the chapter with a few applications of the upper bound axiom.

1.30 Archimedean axiom

Given any real number x, there exists an integer $n \geqslant x$. □

Another way of saying this is to say that the integers are unbounded above. We tacitly assumed 1.30 in our proof of the existence of $\sqrt{2}$ (see 1.27).

In fact, the Archimedean axiom is a consequence of the upper bound axiom since, if the integers were bounded above, then they would have a supremum M. But then there would have to be an integer n such that $n > M - 1$ (since $M - 1 < M$ and therefore cannot be an upper bound). However, this implies that $n + 1 > M$, which contradicts the fact that M is an upper bound.

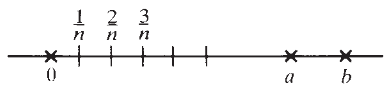
1.31 Density of the rationals

We can use 1.30 to show that, between any two real numbers $a < b$, there exists a rational number r, i.e. $a < r < b$.

All we have to do is to choose an integer n such that

$$n > \frac{1}{b - a},$$

and then walk along the real line starting from 0 with steps of length $1/n$.



It is clear that we must eventually step into the interval (a, b), i.e. we must have

$$a < \frac{m}{n} < b$$

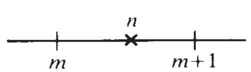
for some integer m. □

We can show similarly that the irrationals are densely distributed on the real line. To show that there is an irrational number between any two real numbers $a < b$, we take a bus to $\sqrt{2}$ and start walking from there. We must bear in mind, however, that the service may be erratic since Route 2 is irrational!

1.32 Theorem

Any non-empty set of integers which is bounded below has a minimum.

Proof Let the set be E, and let $m = \inf E$. We have to show $m \in E$. Now there must exist $n \in E$ such that $n < m + 1$.



Therefore $n - 1 < m$, so there cannot be any integer between n and m. Therefore we must have $n = \min E = m$. □

1.33 Corollary: Mathematical induction

If $P(n)$ is a proposition involving a positive integer n such that $P(1)$ is true, and $P(n)$ true implies $P(n + 1)$ true, then $P(n)$ must be true for all n.

Proof If $P(n)$ is false for any n, then the set

$$E = \{n \geq 1 : P(n) \text{ is false}\}$$

will be non-empty and bounded below. Therefore, by 1.32, E will have a minimum, i.e. there will be a smallest integer n for which $P(n)$ is false. We cannot have $n = 1$, since $P(1)$ is true, and we cannot have $n > 1$, since then we would have $P(n - 1)$ true and $P(n)$ false, which contradicts the induction assumption. We are therefore forced to the conclusion that $P(n)$ is true for all n. □

Observe that any attempt to use 1.23 to prove 1.32 would result in a circular argument since 1.23 was proved by induction!

1.34 Miscellaneous exercises

1. Solve the following inequalities:

   (i) $\dfrac{x + 1}{x + 2} > 3$; (ii) $\dfrac{x + 1}{x + 2} < \dfrac{x + 3}{x + 4}$;

   (iii) $|x + 2| > |3x + 4|$; (iv) $|2x^2 - 11x + 14| < 2$.

2. Prove the following inequalities:
   (i) $(a + b)^2 \leq 2a^2 + 2b^2$;
   (ii) $\sqrt{[(a + b)^2 + (c + d)^2]} \leq \sqrt{(a^2 + c^2)} + \sqrt{(b^2 + d^2)}$;
   (iii) $||a| - |b|| \leq |a \pm b| \leq |a| + |b|$.
3. Prove $2^n > n^3$ for all integers $n \geq 10$.
4. Prove $n! > 2^n$ for all integers $n \geq 4$.
5. Show that a set E is bounded if and only if there exists M such that $|x| \leq M$ for all $x \in E$.
6. Show that if $M = \sup E$, and $\varepsilon > 0$ is given, then there exists $x \in E$ such that $x > M - \varepsilon$.

State and prove a corresponding result for $m = \inf E$.

7. Show that the lower bound axiom is a consquence of the upper bound axiom. *Hint:* Given non-empty E bounded below, let F be the set of all lower bounds of E. Show F is non-empty bounded above, therefore has a supremum which must be the infimum of E.

8. Show $\sqrt{3}$, $\sqrt[3]{4}$ exist as real numbers. *Hint:* Argue as in 1.27.

2 Infinite sequences

Infinitesimal calculus is based on the concept of a limit. Differentiation involves taking the limit of dy/dx as dx tends to zero. The definite integral $\int_a^b f(x)\,dx$ is the limit, as dx tends to zero, of finite sums $\sum f(x)\,dx$. A proper understanding of calculus therefore requires analysis of limiting processes.

We shall analyse limiting processes, in the first instance, in the context of convergence of an infinite sequence of real numbers. Having absorbed the basic idea in this relatively simple situation, we shall then be in a position to appreciate more complicated forms of limit.

By an infinite sequence, we mean an unending succession or progression of numbers, e.g. 1, 2, 3, 4, . . . , or 2, 4, 6, 8, . . . , or 1, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. We shall use the notation $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$ for a general sequence. We shall often abbreviate this to $(a_n)_{n\geqslant 1}$, or simply (a_n), or possibly even just a_n, allowing a little innocent confusion between the full sequence $a_1, a_2, a_3, \ldots$ and its nth term a_n.

It may be that we can give a formula for a_n, e.g., $a_n = n$, or $2n$, or $1/n$. It may be that we can't, e.g. $a_n =$ the height in inches of P.C. n of the Nottinghamshire Constabulary.

A sequence will be said to *converge* (or be *convergent*) if its nth term approaches a definite value as n gets larger and larger. For example, the sequence $(1/n)$ converges because $1/n$ gets closer and closer to 0 as n gets larger and larger. Otherwise a sequence will be said to *diverge* (or be *divergent*). For example, the sequence (n) doesn't settle round any value as n gets large.

If the nth term a_n of the sequence (a_n) approaches the value a as n gets large, we shall say (a_n) *converges to a,* and call a the *limit* of (a_n). So, for example, the sequence $(1/n)$ converges to 0.

Notice, however, that $1/n$ is never actually *equal* to 0. All one can say is that $1/n$ *approximates* to 0 to within any required degree of accuracy if n is taken large enough. For example, if accuracy to within 10^{-6} is required, then n must be taken to be greater than 10^6.

This leads us to make the following definition.